Script **generated by TTT**

Title: Seidl: Programmoptimierung (29.10.2015)

Date: Thu Oct 29 08:33:43 CET 2015

Duration: 89:14 min

Pages: 55

Question:

How do we compute $\mathcal{A}[u]$ for every program point u ??

... which transformations can be composed to the effect of a path $\pi = k_1 \ldots k_r$:

$$[\![\pi]\!]^\sharp = [\![k_r]\!]^\sharp \circ \ldots \circ [\![k_1]\!]^\sharp$$

The effect $[\![k]\!]^\sharp$ of an edge $k = (u, lab, v)$ only depends on the label lab, i.e., $[\![k]\!]^\sharp = [\![lab]\!]^\sharp$ where:

$$\begin{array}{lllll}
[\![;]\!]^\sharp A & = & A & & \\
[\![Pos(e)]\!]^\sharp A & = & [\![Neg(e)]\!]^\sharp A & = & A \cup \{e\} \\
[\![x = e;]\!]^\sharp A & = & (A \cup \{e\}) \backslash Expr_x & & \text{where}
\end{array}$$

$Expr_x$ all expressions which contain x

$$\begin{array}{lll}
[\![x = M[e];]\!]^\sharp A & = & (A \cup \{e\}) \backslash Expr_x \\
[\![M[e_1] = e_2;]\!]^\sharp A & = & A \cup \{e_1, e_2\}
\end{array}$$

Question:

$x + 5y = 7$
$-2x - y = -3$

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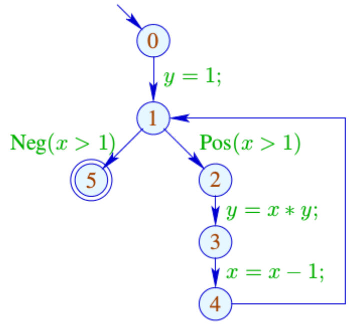
We collect all restrictions to the values of $\mathcal{A}[u]$ into a system of constraints:

$$\begin{array}{lll} \mathcal{A}[start] & \subseteq & \emptyset \\ \mathcal{A}[v] & \subseteq & [\![k]\!]^\sharp(\mathcal{A}[u]) \qquad k = (u, \_, v) \quad \text{edge} \end{array}$$

Wanted:

- a maximally large solution (??)
- an algorithm which computes this :-)

Example:



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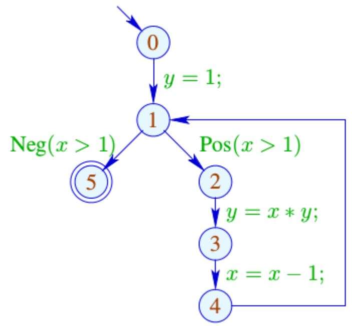

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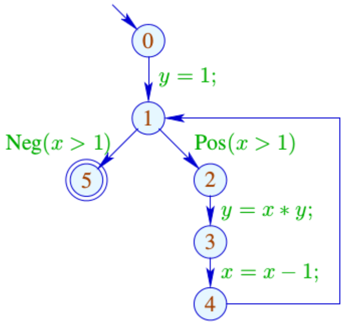

$$\mathcal{A}[0] \subseteq \emptyset$$

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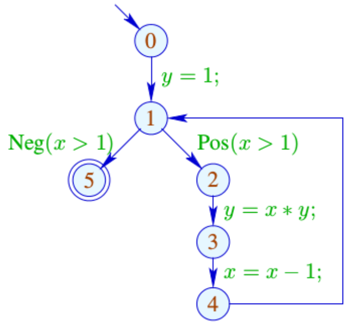

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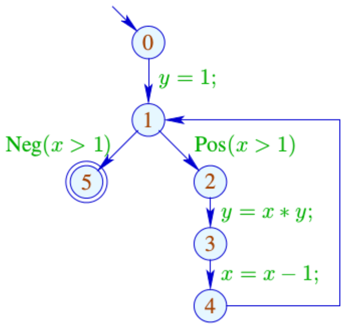

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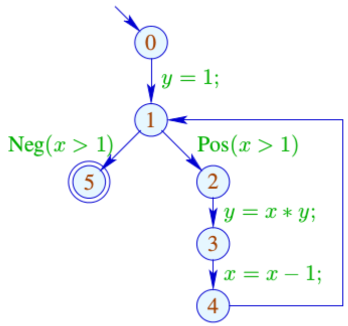

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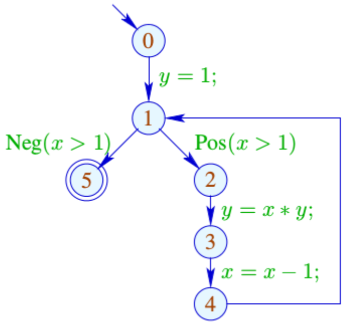

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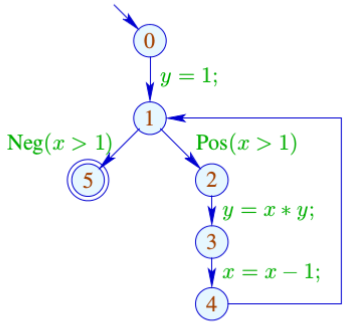

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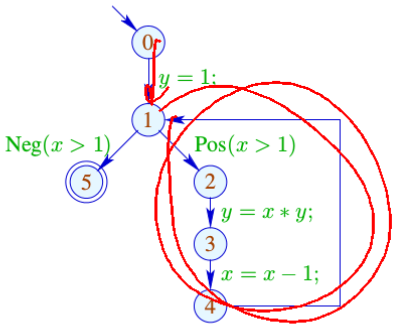

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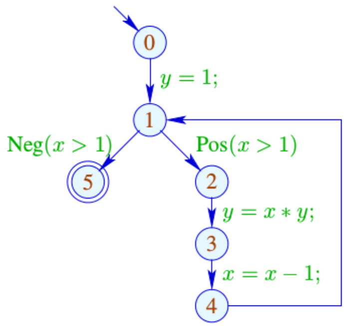

Solution:

$$\begin{array}{lll} \mathcal{A}[0] & = & \emptyset \\ \mathcal{A}[1] & = & \{1\} \\ \mathcal{A}[2] & = & \{1, x > 1\} \\ \mathcal{A}[3] & = & \{1, x > 1\} \\ \mathcal{A}[4] & = & \{1\} \\ \mathcal{A}[5] & = & \{1, x > 1\} \end{array}$$

Observation:

- The possible values for $\mathcal{A}[u]$ form a complete lattice:

$$\mathbb{D} = 2^{Expr} \quad \text{with} \quad B_1 \sqsubseteq B_2 \quad \text{iff} \quad B_1 \supseteq B_2$$

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- The functions $[\![k]\!]^\sharp : \mathbb{D} \to \mathbb{D}$ are monotonic, i.e.,

$$[\![k]\!]^\sharp(B_1) \sqsubseteq [\![k]\!]^\sharp(B_2) \quad \text{whenever} \quad B_1 \sqsubseteq B_2$$

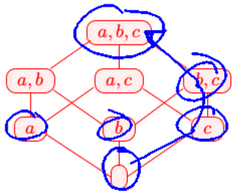
Background 2: Complete Lattices

A set $\mathbb{D}$ together with a relation $\sqsubseteq \subseteq \mathbb{D} \times \mathbb{D}$ is a partial order if for all $a, b, c \in \mathbb{D}$,

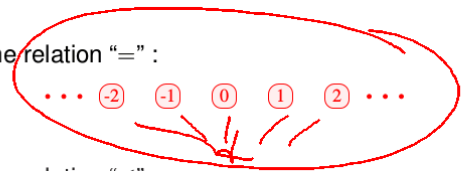
$$\begin{array}{lll} a \sqsubseteq a & & \textit{reflexivity} \\ a \sqsubseteq b \wedge b \sqsubseteq a & \Longrightarrow \quad a = b & \textit{anti-symmetry} \\ a \sqsubseteq b \wedge b \sqsubseteq c & \Longrightarrow \quad a \sqsubseteq c & \textit{transitivity} \end{array}$$

Examples

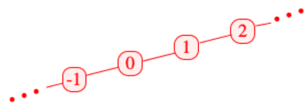
1. $\mathbb{D} = 2^{\{a,b,c\}}$ with the relation "$\subseteq$" :



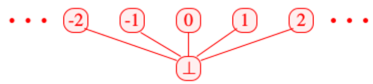
2. $\mathbb{Z}$ with the relation "$=$" :



3. $\mathbb{Z}$ with the relation "$\leq$" :



4. $\mathbb{Z}_\perp = \mathbb{Z} \cup \{\perp\}$ with the ordering:



$d \in \mathbb{D}$ is called upper bound for $X \subseteq \mathbb{D}$ if

$$x \sqsubseteq d \qquad \text{for all } x \in X$$

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Caveat

- $\{0, 2, 4, \ldots\} \subseteq \mathbb{Z}$ has no upper bound!
- $\{0, 2, 4\} \subseteq \mathbb{Z}$ has the upper bounds $4, 5, 6, \ldots$

A complete lattice (cl) $\mathbb{D}$ is a partial ordering where every subset $X \subseteq \mathbb{D}$ has a least upper bound $\bigsqcup X \in \mathbb{D}$.

Remark

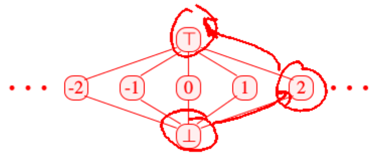
Every complete lattice has

→ a least element $\bot = \bigsqcup \emptyset \in \mathbb{D}$;

→ a greatest element $\top = \bigsqcup \mathbb{D} \in \mathbb{D}$.

Examples

1. $\mathbb{D} = 2^{\{a,b,c\}}$ is a cl.
2. $\mathbb{D} = \mathbb{Z}$ with "$=$" is not.
3. $\mathbb{D} = \mathbb{Z}$ with "$\leq$" is neither.
4. $\mathbb{D} = \mathbb{Z}_\bot$ is also not.
5. With an extra element $\top$, we obtain the flat lattice $\mathbb{Z}_\bot^\top = \mathbb{Z} \cup \{\bot, \top\}$:



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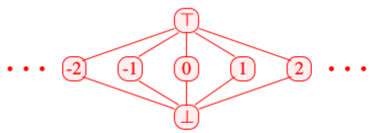

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Proof

Construct $U = \{u \in \mathbb{D} \mid \forall\, x \in X : \; u \sqsubseteq x\}$.

// the set of all lower bounds of X

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// the set of all lower bounds of X

Set: $g := \bigsqcup U$

Claim: $g = \bigsqcap X$

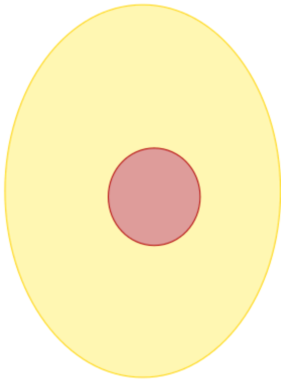
(1) g is a lower bound of X :

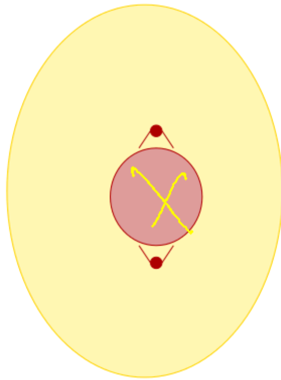
Assume $x \in X$. Then:

$u \sqsubseteq x$ for all $u \in U$

$\Longrightarrow$ x is an upper bound of U

$\Longrightarrow$ $g \sqsubseteq x$





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Constraint for $\mathcal{A}[v]$ $(v \neq start)$:

$$\mathcal{A}[v] \quad \subseteq \quad \bigcap\{[\![k]\!]^\sharp\,(\mathcal{A}[u]) \mid k = (u, \_, v) \text{ edge}\}$$

A mapping $f : \mathbb{D}_1 \to \mathbb{D}_2$ is called monotonic, if $f(a) \sqsubseteq f(b)$ for all $a \sqsubseteq b$.

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Because:

$$x \sqsupseteq d_1 \wedge \ldots \wedge x \sqsupseteq d_k \quad \text{iff} \quad x \sqsupseteq \bigsqcup\{d_1, \ldots, d_k\}$$

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Obviously, every such f is monotonic.

(2) $\mathbb{D}_1 = \mathbb{D}_2 = \mathbb{Z}$ (with the ordering "$\leq$"). Then:

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$f(x) = x \setminus \{1,2\}$

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- $\mathsf{inv}\,x = -x$ is not monotonic.

Theorem

If $f_1 : \mathbb{D}_1 \to \mathbb{D}_2$ and $f_2 : \mathbb{D}_2 \to \mathbb{D}_3$ are monotonic, then also $f_2 \circ f_1 : \mathbb{D}_1 \to \mathbb{D}_3$.

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If $\mathbb{D}_2$ is a complete lattice, then the set $[\mathbb{D}_1 \to \mathbb{D}_2]$ of monotonic functions $f : \mathbb{D}_1 \to \mathbb{D}_2$ is also a complete lattice where

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In particular for $F \subseteq [\mathbb{D}_1 \to \mathbb{D}_2]$,

$$\bigsqcup F = f \quad \text{mit} \quad f\,x = \bigsqcup \{g\,x \mid g \in F\}$$

For functions $f_i\,x = a_i \cap x \cup b_i$, the operations "$\circ$", "$\sqcup$" and "$\sqcap$" can be explicitly defined by:

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(f_2 \circ f_1)\,x &= \boxed{a_1 \cap a_2} \cap x \cup \boxed{a_2 \cap b_1 \cup b_2} \\
(f_1 \sqcup f_2)\,x &= \boxed{(a_1 \cup a_2)} \cap x \cup \boxed{b_1 \cup b_2} \\
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\end{aligned}$$

show that
f is mono

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\end{aligned}
$$

$$
\begin{aligned}
f_2\, (f_1\, x) &= a_2 \cap (a_1 \cap x \cup b_1) \cup b_2 \\
&= \boxed{a_1 \cap a_2} \cap x \cup \boxed{a_2 \cap b_1 \cup b_2}
\end{aligned}
$$

Wanted: minimally small solution for:

$$x_i \sqsupseteq f_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n \qquad (*)$$

where all $f_i : \mathbb{D}^n \to \mathbb{D}$ are monotonic.

For functions $f_i\, x = a_i \cap x \cup b_i$, the operations "$\circ$", "$\sqcup$" and "$\sqcap$" can be explicitly defined by:

$$
\begin{aligned}
(f_2 \circ f_1)\, x &= \boxed{a_1 \cap a_2} \cap x \cup \boxed{a_2 \cap b_1 \cup b_2} \\
(f_1 \sqcup f_2)\, x &= \boxed{(a_1 \cup a_2)} \cap x \cup \boxed{b_1 \cup b_2} \\
(f_1 \sqcap f_2)\, x &= \boxed{(a_1 \cup b_1) \cap (a_2 \cup b_2)} \cap x \cup \boxed{b_1 \cap b_2}
\end{aligned}
$$

Wanted: minimally small solution for:

$$x_i \sqsupseteq f_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n \qquad (*)$$

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Idea

- Consider $F : \mathbb{D}^n \to \mathbb{D}^n$ where

  $F(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$ with $y_i = f_i(x_1, \ldots, x_n)$.

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$x \sqsupseteq F x$

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- If all f_i are monotonic, then also F.
- We successively approximate a solution. We construct:

$$\bot, \quad F\,\bot, \quad F^2\,\bot, \quad F^3\,\bot, \quad \ldots$$

Hope: We eventually reach a solution ... ???