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Title: Seidl: Programmoptimierung (22.01.2014)

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Discussion:

- Integer Linear Programming (ILP) can decide satisfiability of a finite set of equations/inequations over $\mathbb{Z}$ of the form:

$$\sum_{i=1}^{n} a_i \cdot x_i = b \quad \text{bzw.} \quad \sum_{i=1}^{n} a_i \cdot x_i \geq b\,, \qquad a_i \in \mathbb{Z}$$

- Moreover, a (linear) cost function can be optimized :-)
- Warning: The decision problem is in general, already NP-hard !!!
- Notwithstanding that, surprisingly efficient implementations exist.
- Not just loop fusion, but also other re-organizations of loops yield ILP problems ...

Background 5: Presburger Arithmetic

Many problems in computer science can be formulated without multiplication :-)

Let us first consider two simple special cases ...

1. Linear Equations

$$\exists x, y, z . \quad \begin{array}{rcrcrcr} 2x & + & 3y & & & = & 24 \quad \wedge \\ x & - & y & + & 5z & = & 3 \end{array}$$

Question:

- Is there a solution over $\mathbb{Q}$? P
- Is there a solution over $\mathbb{Z}$?
- Is there a solution over $\mathbb{N}$? NP

Let us reconsider the equations:

$$\begin{array}{rcrcrcr} 2x & + & 3y & & & = & 24 \\ x & - & y & + & 5z & = & 3 \end{array}$$

Answers:

- Is there a solution over $\mathbb{Q}$? Yes
- Is there a solution over $\mathbb{Z}$? No
- Is there a solution over $\mathbb{N}$? No

Complexity:

- Is there a solution over $\mathbb{Q}$? Polynomial
- Is there a solution over $\mathbb{Z}$? Polynomial
- Is there a solution over $\mathbb{N}$? NP-hard

Question:

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$ax = b$

Solution Method for Integers:

Observation 1:

$$a_1x_1 + \ldots + a_kx_k = b \qquad (\forall\, i:\ a_i \neq 0)$$

has a solution iff

$$\mathrm{gcd}\{a_1, \ldots, a_k\} \quad | \quad b$$

$\mathrm{gcd}(a_1, a_2) = d$

$a_1x_1 + a_2x_2 = d$

Example:

$$5y - 10z = 18$$

has no solution over $\mathbb{Z}$:-)

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Observation 2:

Adding a multiple of one equation to another does not change the set of solutions :-)

Example:

$$\begin{array}{rcrcrcr} 2x & + & 3y & & & = & 24 \\ x & - & y & + & 5z & = & 3 \end{array}$$

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$$\Longrightarrow$$

$$\begin{array}{rcrcrcr} & & 5y & - & 10z & = & 18 \\ x & - & y & + & 5z & = & 3 \end{array}$$

Observation 3:

Adding multiples of columns to another column is an invertible transformation which we keep track of in a separate matrix ...

$$\begin{array}{ccc|ccccccc} 1 & 0 & 0 & & & 5y & - & 10z & = & 18 \\ 0 & 1 & 0 & x & - & y & + & 5z & = & 3 \\ 0 & 0 & 1 & & & & & & & \end{array}$$

$$\Longrightarrow$$

$$\begin{array}{ccc|ccccccc} 1 & 0 & 0 & & & 5y & & & = & 18 \\ 0 & 1 & 2 & x & - & y & + & 3z & = & 3 \\ 0 & 0 & 1 & & & & & & & \end{array}$$

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$2x$ $2x$ 0

$3x$ x x

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$$\Longrightarrow$$

$$\begin{array}{ccc|ccccccc} 1 & 0 & -3 & & & 5y & & & = & 18 \\ 0 & 1 & 2 & x & - & y & & & = & 3 \\ 0 & 0 & 1 & & & & & & & \end{array}$$

$\Longrightarrow$ triangular form !!

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1&0&0& & & 5y & & & = & 18\\
0&1&2& x & - & y & + & 3z & = & 3\\
0&0&1& & & & & & &
\end{array}$$

$$\Longrightarrow$$

$$\begin{array}{rrr|rcrcrcr}
1&0&-3& & & 5y & & & = & 18\\
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0&1&2& x & - & y & + & 3z & = & 3\\
0&0&1& & & & & & &
\end{array}$$

Observation 4:

- A special solution of a triangular system can be directly read off :-)
- All solutions of a homogeneous triangular system can be directly read off :-)
- All solutions of the original system can be recovered from the solutions of the triangular system by means of the accumulated transformation matrix:-))

Example

$$\begin{array}{rrr|rcrcr} 1 & 0 & -3 & & & 5y & = & 15 \\ 0 & 1 & 2 & x & - & y & = & 3 \\ 0 & 0 & 1 & & & & & \end{array}$$

$y = 3$
$x = 6$

One special solution:

$$[6, 3, 0]^\top$$

All solutions of the homogeneous system are spanned by:

$$[0, 0, 1]^\top$$

Example

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Solving over $\mathbb{N}$

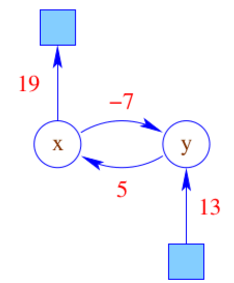
- ... is of major practical importance;
- ... has led to the development of many new techniques;
- ... easily allows to encode NP-hard problems;
- ... remains difficult if just three variables are allowed per equation.

2. One Polynomial Special Case:

$$\begin{array}{rcl} x & \geq & y+5 \\ 19 \geq & x & \\ y & \geq & 13 \\ y & \geq & x-7 \end{array}$$

- There are at most 2 variables per in-equation;
- no scaling factors.

Idea: Represent the system by a graph:



Jean Baptiste Joseph Fourier, 1768–1830

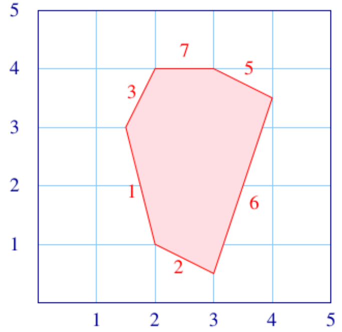
3. A General Solution Method: for $\mathbb{Q}$

Idea: Fourier-Motzkin Elimination

- Successively remove individual variables x !
- All in-equations with positive occurrences of x yield lower bounds.
- All in-equations with negative occurrences of x yield upper bounds.
- All lower bounds must be at most as big as all upper bounds ;-))

Example:

$$
\begin{array}{rcll}
9 & \leq & 4x_1 + x_2 & (1) \\
4 & \leq & x_1 + 2x_2 & (2) \\
0 & \leq & 2x_1 - x_2 & (3) \\
6 & \leq & x_1 + 6x_2 & (4) \\
-11 & \leq & -x_1 - 2x_2 & (5) \\
-17 & \leq & -6x_1 + 2x_2 & (6) \\
-4 & \leq & -x_2 & (7)
\end{array}
$$



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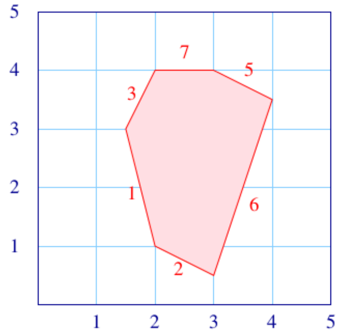

For x_1 we obtain:

$$
\begin{array}{rcll}
9 & \leq & 4x_1 + x_2 & (1) \\
4 & \leq & x_1 + 2x_2 & (2) \\
0 & \leq & 2x_1 - x_2 & (3) \\
6 & \leq & x_1 + 6x_2 & (4) \\
-11 & \leq & -x_1 - 2x_2 & (5) \\
-17 & \leq & -6x_1 + 2x_2 & (6) \\
-4 & \leq & -x_2 & (7)
\end{array}
\qquad
\begin{array}{rcll}
\frac{9}{4} - \frac{1}{4}x_2 & \leq & x_1 & (1) \\
4 - 2x_2 & \leq & x_1 & (2) \\
\frac{1}{2}x_2 & \leq & x_1 & (3) \\
6 - 6x_2 & \leq & x_1 & (4) \\
x_1 & \leq & 11 - 2x_2 & (5) \\
x_1 & \leq & \frac{17}{6} + \frac{1}{3}x_2 & (6) \\
-4 & \leq & -x_2 & (7)
\end{array}
$$

If such an x_1 exists, all lower bounds must be bounded by all upper bounds, i.e.,

$$
\begin{array}{rcll}
\frac{9}{4} - \frac{1}{4}x_2 & \leq & 11 - 2x_2 & (1,5) \\
\frac{9}{4} - \frac{1}{4}x_2 & \leq & \frac{17}{6} + \frac{1}{3}x_2 & (1,6) \\
4 - 2x_2 & \leq & 11 - 2x_2 & (2,5) \\
4 - 2x_2 & \leq & \frac{17}{6} + \frac{1}{3}x_2 & (2,6) \\
\frac{1}{2}x_2 & \leq & 11 - 2x_2 & (3,5) \\
\frac{1}{2}x_2 & \leq & \frac{17}{6} + \frac{1}{3}x_2 & (3,6) \\
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-4 & \leq & -x_2 & (7)
\end{array}
\quad \text{or} \quad
\begin{array}{rcll}
-35 & \leq & -7x_2 & (1,5) \\
-\frac{7}{12} & \leq & \frac{7}{12}x_2 & (1,6) \\
\cancel{-7} & \cancel{\leq} & \cancel{0} & (2,5) \\
\frac{7}{6} & \leq & \frac{7}{3}x_2 & (2,6) \\
-22 & \leq & -5x_2 & (3,5) \\
-\frac{17}{3} & \leq & -\frac{1}{6}x_2 & (3,6) \\
-5 & \leq & 4x_2 & (4,5) \\
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-4 & \leq & -x_2 & (7)
\end{array}
\quad \text{or} \quad
\begin{array}{rcll}
-5 & \leq & -x_2 & (1,5) \\
-1 & \leq & x_2 & (1,6) \\
-7 & \leq & 0 & (2,5) \\
\frac{1}{2} & \leq & x_2 & (2,6) \\
-\frac{22}{5} & \leq & -x_2 & (3,5) \\
-17 & \leq & -x_2 & (3,6) \\
-\frac{5}{4} & \leq & x_2 & (4,5) \\
\frac{1}{2} & \leq & x_2 & (4,6) \\
-4 & \leq & -x_2 & (7)
\end{array}
$$

This is the one-variable case which we can solve exactly:

Example:

$$
\begin{array}{rcll}
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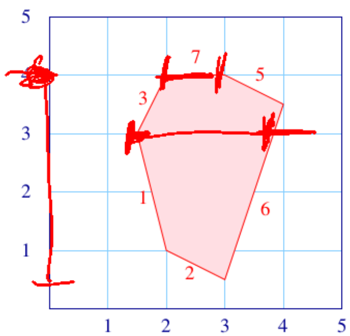

$$\max\left\{-1, \boxed{\tfrac{1}{2}}, -\tfrac{5}{4}, \tfrac{1}{2}\right\} \;\leq\; x_2 \;\leq\; \min\left\{5, \tfrac{22}{5}, 17, \boxed{4}\right\}$$

From which we conclude: $x_2 \in [\frac{1}{2}, 4]$:-)

In General:

- The original system has a solution over $\mathbb{Q}$ iff the system after elimination of one variable has a solution over $\mathbb{Q}$:-)
- Every elimination step may square the number of in-equations
  $\Longrightarrow$ exponential run-time :-((
- It can be modified such that it also decides satisfiability over $\mathbb{Z}$
  $\Longrightarrow$ Omega Test

$$
\begin{array}{rcll}
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William Worthington Pugh, Jr.
University of Maryland, College Park

Idea:

- We successively remove variables. Thereby we omit division ...
- If x only occurs with coefficient ± 1, we apply Fourier-Motzkin elimination :-)
- Otherwise, we provide a bound for a positive multiple of x ...

Consider, e.g., (1) and (6) :

$$
\begin{array}{rcl}
6 \cdot x_1 & \leq & 17 + 2x_2 \\
9 - x_2 & \leq & 4 \cdot x_1
\end{array}
$$

$$
\max \{-1, \boxed{\tfrac{1}{2}}, -\tfrac{5}{4}, \tfrac{1}{2}\} \quad \leq \quad x_2 \quad \leq \quad \min \{5, \tfrac{22}{5}, 17, \boxed{4}\}
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Consider, e.g., (1) and (6) :

$$
\begin{array}{rcl}
6 \cdot x_1 & \leq & 17 + 2x_2 \\
9 - x_2 & \leq & 4 \cdot x_1
\end{array}
$$

W.l.o.g., we only consider strict in-equations:

$$
\begin{array}{rcl}
6 \cdot x_1 & < & 18 + 2x_2 \\
8 - x_2 & < & 4 \cdot x_1
\end{array}
$$

... where we always divide by gcds:

$$
\begin{array}{rcl}
3 \cdot x_1 & < & 9 + x_2 \\
8 - x_2 & < & 4 \cdot x_1
\end{array}
$$

This implies:

$$3 \cdot (8 - x_2) \quad < \quad 4 \cdot (9 + x_2)$$

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718

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We thereby obtain:

- If one derived in-equation is unsatisfiable, then also the overall system :-)
- If all derived in-equations are satisfiable, then there is a solution which, however, need not be integer :-(
- An integer solution is guaranteed to exist if there is sufficient separation between lower and upper bound ...
- Assume $\alpha < a \cdot x \qquad b \cdot x < \beta$.

  Then it should hold that:

  $$b \cdot \alpha < a \cdot \beta$$

  and moreover:

  $$\boxed{a \cdot b} < a \cdot \beta - b \cdot \alpha$$

W.l.o.g., we only consider strict in-equations:

$$6 \cdot x_1 \quad < \quad 18 + 2x_2$$
$$8 - x_2 \quad < \quad 4 \cdot x_1$$

... where we always divide by gcds:

$$3 \cdot x_1 \quad < \quad 9 + x_2$$
$$8 - x_2 \quad < \quad 4 \cdot x_1$$

This implies:

$$3 \cdot (8 - x_2) \quad < \quad 4 \cdot (9 + x_2)$$

We thereby obtain:

- If one derived in-equation is unsatisfiable, then also the overall system :-)
- If all derived in-equations are satisfiable, then there is a solution which, however, need not be integer :-(
- An integer solution is guaranteed to exist if there is sufficient separation between lower and upper bound ...
- Assume $\alpha < a \cdot x \qquad b \cdot x < \beta$.

  Then it should hold that:

  $$b \cdot \alpha < a \cdot \beta$$

  and moreover:

  $$\boxed{a \cdot b} < a \cdot \beta - b \cdot \alpha$$

... in the Example:

$$12 \quad < \quad 4 \cdot (9 + x_2) - 3 \cdot (8 - x_2)$$

or:

$$12 \quad < \quad 12 + 7x_2$$

or:

$$0 \quad < \quad x_2$$

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Discussion:

- If the strengthened in-equations are satisfiable, then also the original system. The reverse implication may be wrong :-(
- In the case where upper and lower bound are not sufficiently separated, we have:

$$a \cdot \beta \leq b \cdot \alpha + \boxed{a \cdot b}$$

or:

$$b \cdot \alpha < ab \cdot x < b \cdot \alpha + \boxed{a \cdot b}$$

Division with b yields:

$$\alpha < a \cdot x < \alpha + \boxed{a}$$

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alt ergo

Discussion (cont.):

z3

→ Fourier-Motzkin Elimination is not the best method for rational systems of in-equations.

→ The Omega test is necessarily exponential :-)

If the system is solvable, the test generally terminates rapidly.

It may have problems with unsolvable systems :-(

→ Also for ILP, there are other/smarter algorithms ...

→ For programming language problems, however, it seems to behave quite well :-)

4. Generalization to a Logic

Disjunction:

$$\begin{array}{lll}(x - 2y = 15 & \wedge \quad x + y = 7) & \vee \\ (x + y = 6 & \wedge \quad 3x + z = -8) & \end{array}$$

Quantors:

$$\neg \left(\exists x : \quad z - 2x = 42 \quad \wedge \quad z + x = 19 \right)$$

$$= \forall x : z - 2x \neq 42 \vee z + x \neq 19$$

Mojzesz Presburger, 1904–1943 (?)

Presburger Arithmetic == full arithmetic without multiplication

Arithmetic : highly undecidable :-(
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$\Longrightarrow$ Hilbert's 10th Problem

$\Longrightarrow$ Gödel's Theorem

$\exists x_1, \ldots, x_k .\ p(x_1, \ldots, x_k) = 0$