Script generated by TTT

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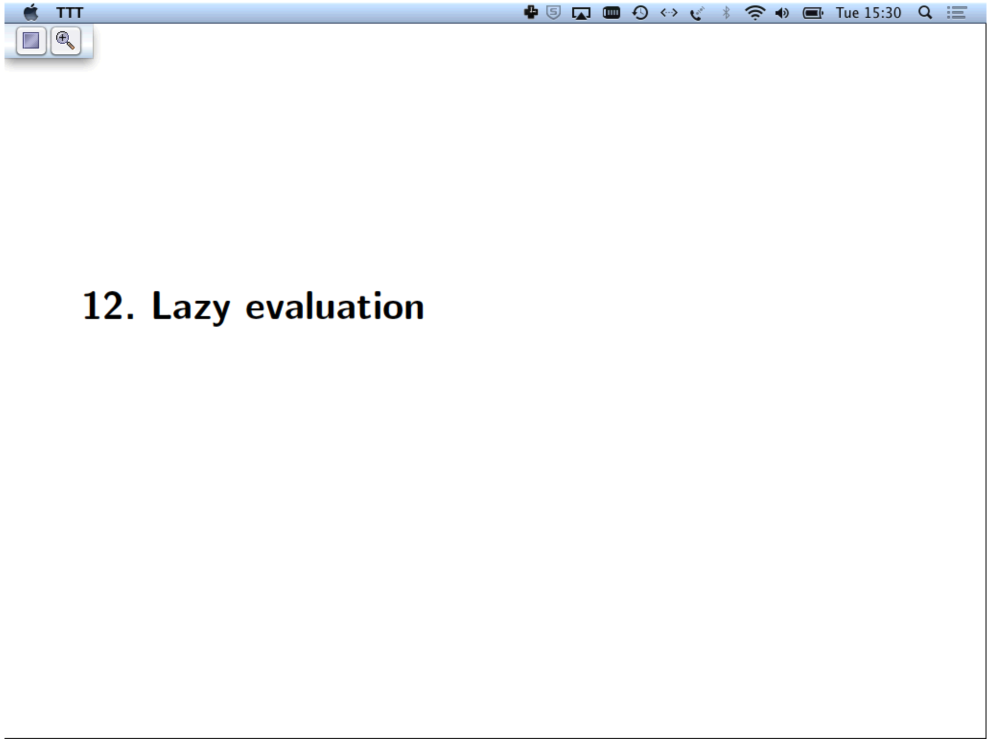
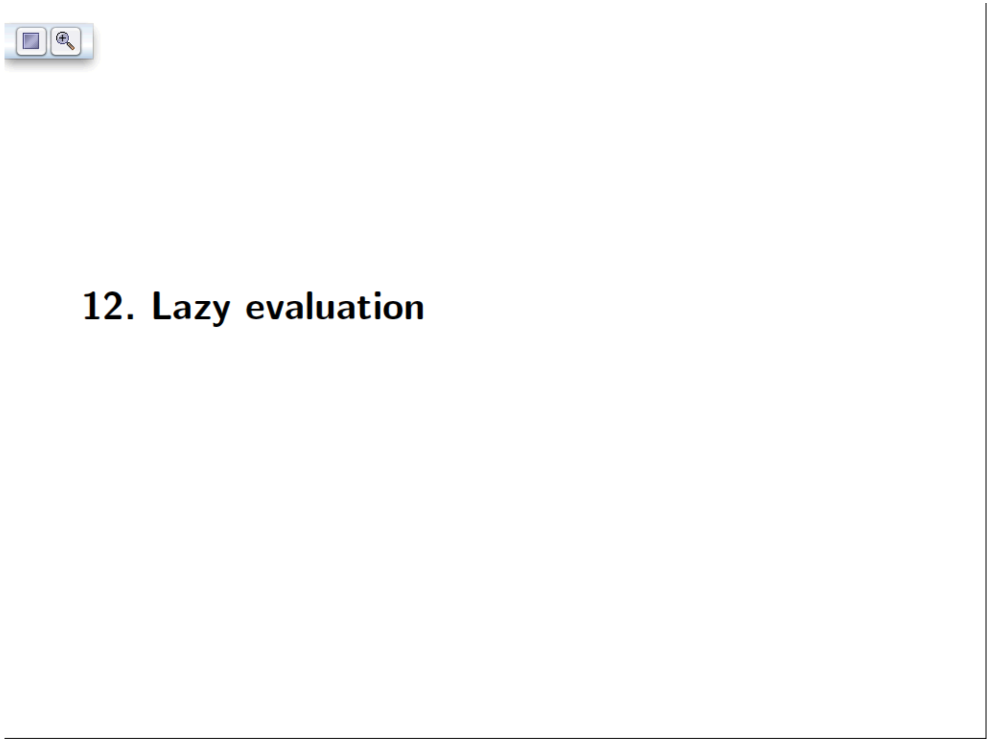


12. Lazy evaluation



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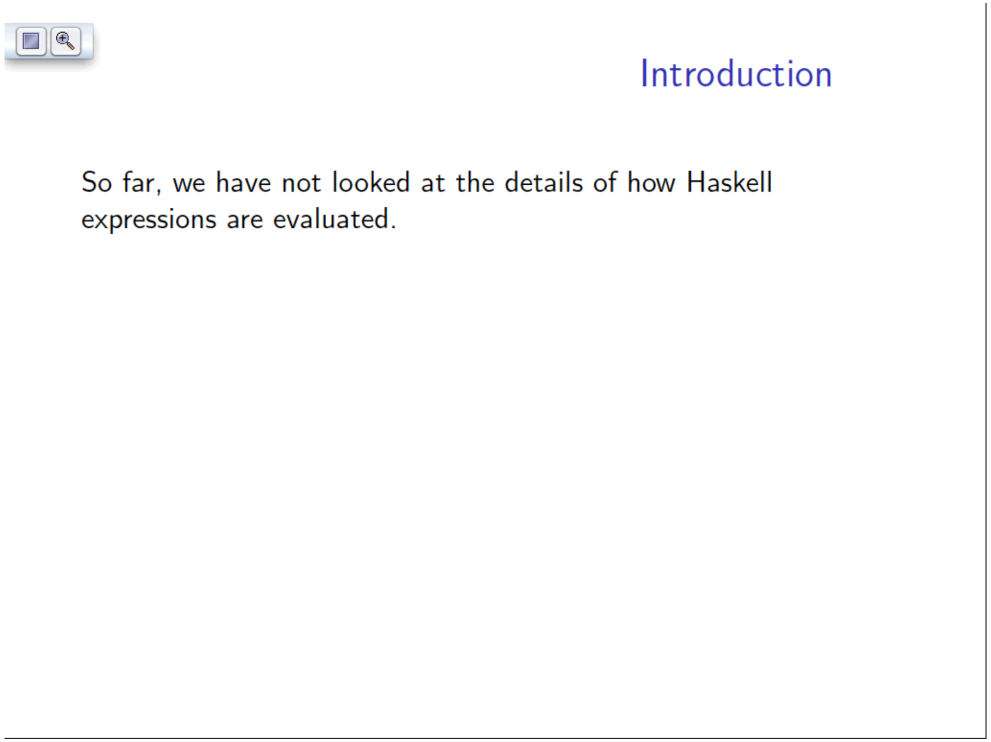

Introduction

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- Terminates as often as possible
- Supports infinite lists
- Increases modularity

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Therefore Haskell is called a *lazy functional langauge*.

Evaluating expressions

Expressions are evaluated (*reduced*) by successively applying definitions until no further reduction is possible.

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Example:

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sq :: Integer -> Integer
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One evaluation:

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```
sq (3+4)  =  (3+4) * (3+4)
```

Comparison: termination

Definition:

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loop = tail loop
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Theorem If expression e has a terminating reduction sequence, then outermost reduction of e also terminates.

Outermost reduction terminates as often as possible

Why is this useful?

Example

Can build your own control constructs:

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switch :: Int -> a -> a -> a
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fac :: Int -> Int
fac n = switch n (n * fac(n-1)) 1
```

Comparison: Number of steps

Innermost reduction:

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sq (3+4) =  sq 7  =  7 * 7  =  49
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More outermost than innermost steps!

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Outermost reduction:

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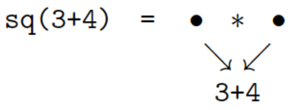
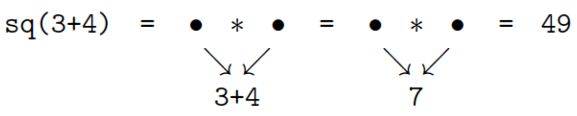
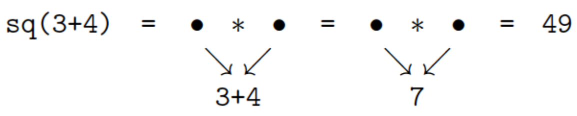
More outermost than innermost steps!
How can outermost reduction be improved?
Sharing!





The expression 3+4 is only evaluated *once!*



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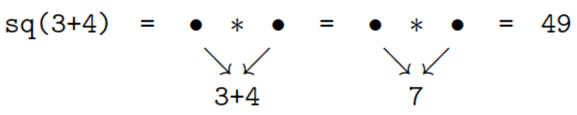
Lazy evaluation := outermost reduction + sharing



The expression 3+4 is only evaluated *once!*

Lazy evaluation := outermost reduction + sharing

Theorem
Lazy evaluation never needs more steps than innermost reduction.

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- Each argument is evaluated at most once (sharing!)

Pattern matching

Example

```
f :: [Int] -> [Int] -> Int
f []     ys     =  0
f (x:xs) []     =  0
f (x:xs) (y:ys) =  x+y
```

Pattern matching

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f [1..3] [7..9]
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f [1..3] [7..9]                      -- does f.1 match?
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f [1..3] [7..9]                  -- does f.1 match?
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=  f (1 : [2..3]) (7 : [8..9])   -- does f.3 match?
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=  f (1 : [2..3]) (7 : [8..9])   -- does f.3 match?
=  1+7
=  8
```

Guards

Example

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f m n p | m >= n && m >= p  =  m
        | n >= m && n >= p  =  n
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f (2+3) (4-1) (3+9)
  ?  2+3 >= 4-1 && 2+3 >= 3+9
  ?  =  5 >= 3 && 5 >= 3+9
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f (2+3) (4-1) (3+9)
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  ?  =  5 >= 3 && 5 >= 3+9
  ?  =  True    && 5 >= 3+9
  ?  =  5 >= 3+9
  ?  =  5 >= 12
  ?  =  False
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  ?  =  False
  ?  otherwise = True
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  ? 3 >= 5 && 3 >= 12
  ?  =  False && 3 >= 12
  ?  =  False
  ? otherwise = True
=  12
```

`where`

Same principle: definitions in `where` clauses are only evaluated when needed and only as much as needed.

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(\x -> False && x) True  =  False && True  =  False
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Built-in functions

Arithmetic operators and other built-in functions
evaluate their arguments first

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Arithmetic operators and other built-in functions
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Example

`3 * 5` is a redex

`0 * head (...)` is not a redex

Predefined functions from Prelude

They behave like their Haskell definition:

```
(&&) :: Bool -> Bool -> Bool
True  && y  =  y
False && y  =  False
```

Slogan

Lazy evaluation evaluates an expression only when needed and only as much as needed.

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(*“Call by need”*)

12.1 Applications of lazy evaluation

Minimum of a list

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min  =  head . inSort
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inSort :: Ord a => [a] -> [a]
inSort []       =  []
inSort (x:xs)   =  ins x (inSort xs)
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ins :: Ord a => a -> [a] -> [a]
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ins x (y:ys) | x <= y      =  x : y : ys
             | otherwise   =  y : ins x ys
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⟹  inSort [6,1,7,5]
    =  ins 6 (ins 1 (ins 7 (ins 5 [])))
```

```
min [6,1,7,5]   =  head(inSort [6,1,7,5])
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min [6,1,7,5]  =  head(inSort [6,1,7,5])
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min [6,1,7,5]  =  head(inSort [6,1,7,5])
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=  head(ins 6 (ins 1 (ins 7 (5 : []))))
```

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min [6,1,7,5]  =  head(inSort [6,1,7,5])
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min [6,1,7,5]  =  head(inSort [6,1,7,5])
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=  head(1 : ins 6 (5 : ins 7 []))
=  1
```

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min [6,1,7,5]  =  head(inSort [6,1,7,5])
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Lazy evaluation needs only linear time

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min [6,1,7,5]  =  head(inSort [6,1,7,5])
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although `inSort` is quadratic

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```

Lazy evaluation needs only linear time
although `inSort` is quadratic
because the sorted list is never constructed completely

Warning: this depends on the exact algorithm and does not work so nicely with all sorting functions!

Maximum of a list

```
max  =  last . inSort
```

Complexity?

12.2 Infinite lists

Example

A recursive definition

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ones :: [Int]
ones  =  1 : ones
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that defines an infinite list of 1s:

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ones
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```
ones  =  1 : ones
```

Example

A recursive definition

```
ones :: [Int]
ones  =  1 : ones
```

that defines an infinite list of 1s:

```
ones  =  1 : ones  =  1 : 1 : ones  =  ...
```

What GHCi has to say about it:

```
> ones
[1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1
```

Haskell lists can be finite or infinite

But Haskell can compute with infinite lists, thanks to lazy evaluation:

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```
> head ones
1
```

Remember:

> Lazy evaluation evaluates an expression only as much as needed

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Outermost reduction: `head ones   =   head (1 : ones)   =   1`

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Outermost reduction: `head ones   =   head (1 : ones)   =   1`

Innermost reduction:

```
head ones
=  head (1 : ones)
=  head (1 : 1 : ones)
=  ...
```

Haskell lists are never actually infinite but only potentially infinite

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Lazy evaluation computes as much of the infinite list as needed

This is how partially evaluated lists are represented internally:

1 : 2 : 3 : $\boxed{\text{code pointer to compute rest}}$

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In general: finite prefix followed by code pointer

Why (potentially) infinite lists?

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- They come for free with lazy evaluation
- They increase modularity:
  list producer does not need to know
  how much of the list the consumer wants

Example: The sieve of Eratosthenes

1. Create the list 2, 3, 4, . . .

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2 3 4 5 6 7 8 9 10 11 12 . . .

In Haskell:

```haskell
primes :: [Int]
primes  =  sieve [2..]

sieve :: [Int] -> [Int]
sieve (p:xs)   =  p : sieve [x | x <- xs,
```

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primes  =  sieve [2..]  =  sieve (2:[3..])
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=  2 : 3 : sieve [x | x <- [x|x <- [4..],  x ‘mod‘ 2 /= 0]
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= ...
```

Modularity!

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The first 10 primes:

```
> take 10 primes
[2,3,5,7,11,13,17,19,23,29]
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The primes between 100 and 150:

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> takeWhile (<150) (dropWhile (<100) primes)
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> [(p,q) | (p,q) <-                         , p+2==q]
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> [(p,q) | (p,q) <- zip primes (tail primes), p+2==q]
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Primality test?

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> 101 `elem` primes
True

> 102 `elem` primes
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Sharing!

There is only one copy of `primes`

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Sharing!

There is only one copy of `primes`

Every time part of `primes` needs to be evaluated

Example: when computing `take 5 primes`

`primes` is (invisibly!) updated to remember the evaluated part

Example: `primes = 2 : 3 : 5 : 7 : 11 : sieve ...`

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Nothing special, just the automatic result of sharing

The list of Fibonacci numbers

Idea: 0 1 1 2 ...

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fibs :: [Integer]
fibs = 0 : 1 :
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fibs :: [Integer]
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```

How about

```
fibs = 0 : 1 : [x+y | x <- fibs, y <- tail fibs]
```

Hamming numbers

Game tree

```
data Tree p v = Tree p v [Tree p v]
```

Separates move computation and valuation from move selection

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gameTree next val = tree where
  tree p = Tree p (val p) (map tree (next p))
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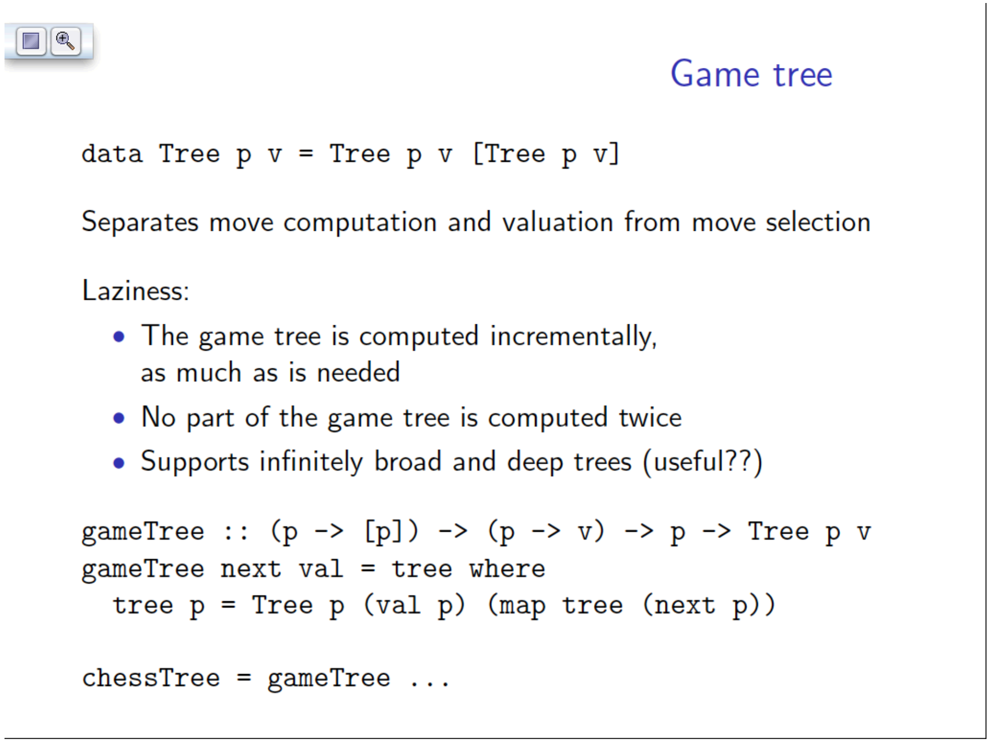

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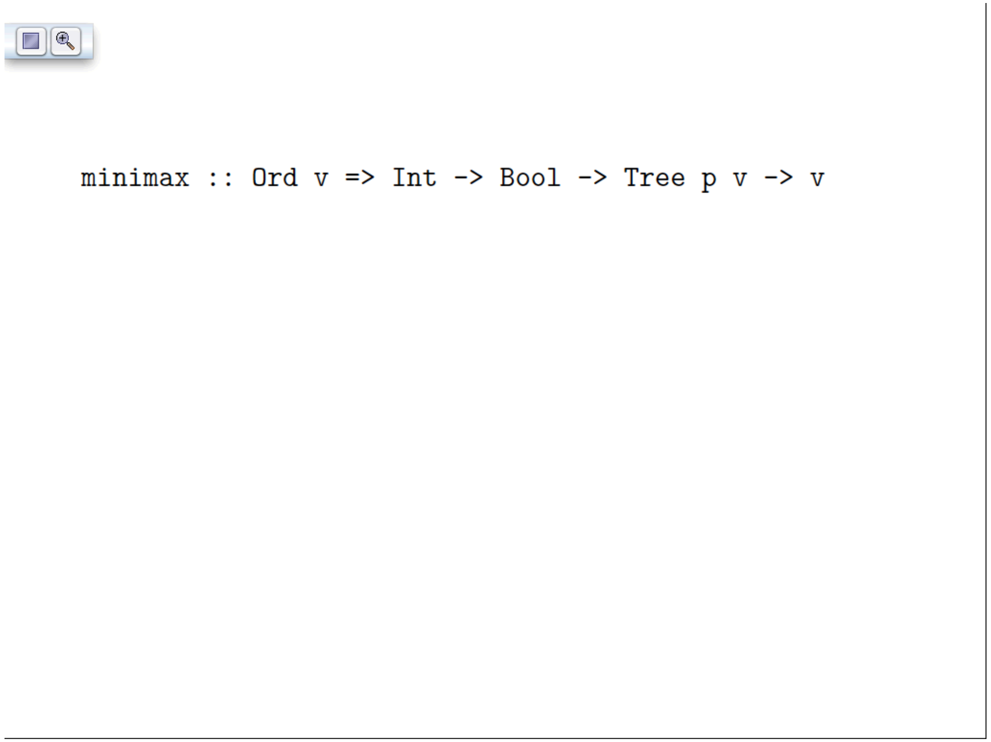
chessTree = gameTree ...
```



```
minimax :: Ord v => Int -> Bool -> Tree p v -> v
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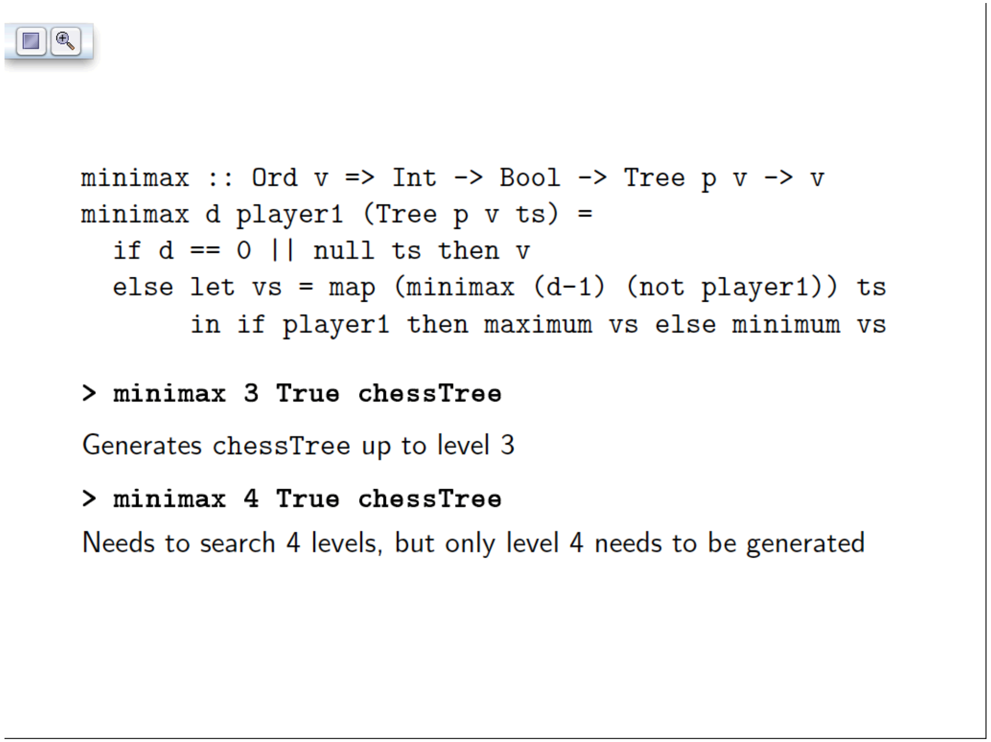


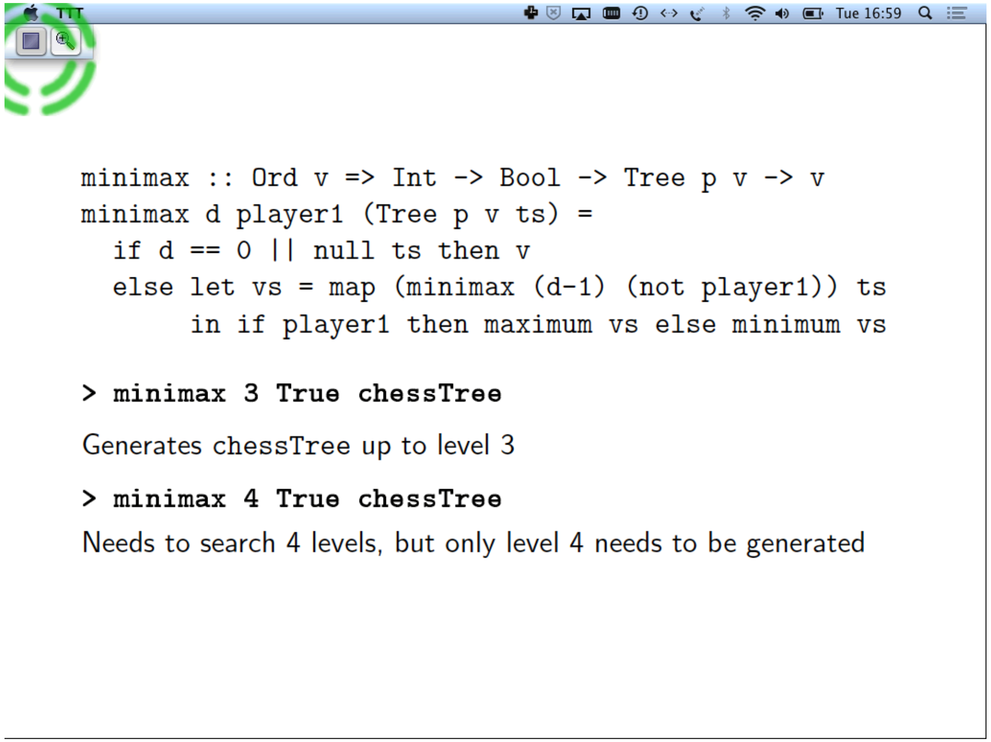
```
minimax :: Ord v => Int -> Bool -> Tree p v -> v
minimax d player1 (Tree p v ts) =
  if d == 0 || null ts then v
  else let vs = map (minimax (d-1) (not player1)) ts
       in if player1 then maximum vs else minimum vs
```

> minimax 3 True chessTree

Generates chessTree up to level 3

> minimax 4 True chessTree

Needs to search 4 levels, but only level 4 needs to be generated



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