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One by one (Union)

Let $c(x)$ = cost of adding 1 element to set of size x

Cost of adding m elements to a set of n elements:

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If $c(x) = \log_2 x \Longrightarrow$
Cost $= O(m * \log_2(n + m)) = O(m * \log_2 n)$

Similar for intersection and difference.

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  A parallel divide and conquer approach

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- Cost: $\Theta(m * \log_2(\frac{n}{m} + 1))$
- Works for many kinds of balanced trees
- For ease of presentation: use concrete type $tree$

Uniform $tree$ type

Red-Black trees, AVL trees, weight-balanced trees, etc can all be implemented with $'b$-augmented trees:

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$('a \times 'b)\ tree$

We work with this type of trees without committing to any particular kind of balancing schema.

In this chapter: $tree$ abbreviates $('a \times 'b)\ tree$

Just *join*

Can synthesize all BST interface functions from just one function:

$join\ l\ a\ r$

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Thus *join* determines the balancing schema

Just *join*

Given $join :: tree \Rightarrow {}'a \Rightarrow tree \Rightarrow tree$
we implement

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Given $join :: tree \Rightarrow {}'a \Rightarrow tree \Rightarrow tree$
we implement

$insert :: {}'a \Rightarrow tree \Rightarrow tree$
$delete :: {}'a \Rightarrow tree \Rightarrow tree$
$union :: tree \Rightarrow tree \Rightarrow tree$
$inter :: tree \Rightarrow tree \Rightarrow tree$
$diff :: tree \Rightarrow tree \Rightarrow tree$

$split :: tree \Rightarrow {}'a \Rightarrow tree \times bool \times tree$

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$insert$ :: $'a \Rightarrow tree \Rightarrow tree$

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$union$ :: $tree \Rightarrow tree \Rightarrow tree$

$union\ t_1\ t_2 =$
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in $join\ l'\ a\ r'$)

$inter :: tree \Rightarrow tree \Rightarrow tree$

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in

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in if ain then $join\ l'\ a\ r'$

$diff$ :: $tree \Rightarrow tree \Rightarrow tree$

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```
diff t1 t2 =
(if t1 = ⟨⟩ then ⟨⟩ else
 if t2 = ⟨⟩ then t1 else
 case t2 of
   ⟨l2, (a, _), r2⟩ ⇒
     let (l1, _, r1) = split t1 a;
         l' = diff l1 l2;
         r' = diff r1 r2
     in join2 l' r')
```


$diff :: tree \Rightarrow tree \Rightarrow tree$

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Why this way around: t_1/t_2?

13 Union, Intersection, Difference on BSTs
- Implementation
- Correctness
- Join for Red-Black Trees

Specification of *join*

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Locale context for def of *union* etc

Specification of *union*, *inter*, *diff*

ADT/Locale *Set2* = extension of locale *Set* with

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Specification of *join* and *inv*

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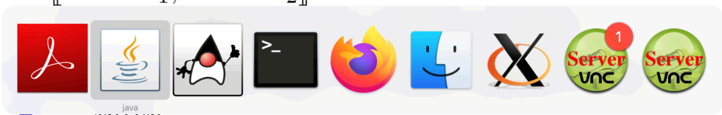

- $\dots\ inter\ \dots$
- $\dots\ diff\ \dots$

Correctness lemmas for *union* etc code

In the context of *join* specification:

- $bst\ t_2 \Longrightarrow$
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Specification of *union*, *inter*, *diff*

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  $\Longrightarrow set\ (union\ s_1\ s_2) = set\ s_1 \cup set\ s_2$
- $[\![invar\ s_1;\ invar\ s_2]\!] \Longrightarrow invar\ (union\ s_1\ s_2)$
- $\dots\ inter\ \dots$
- $\dots\ diff\ \dots$

We focus on *union*.

See `HOL/Data_Structures/Set_Specs.thy`

Correctness lemmas for *union* etc code

In the context of *join* specification:

- $bst\ t_2 \Longrightarrow$
  $set\_tree\ (union\ t_1\ t_2) = set\_tree\ t_1 \cup set\_tree\ t_2$
- $[\![bst\ t_1;\ bst\ t_2]\!] \Longrightarrow bst\ (union\ t_1\ t_2)$
- $[\![inv\ t_1;\ inv\ t_2]\!] \Longrightarrow inv\ (union\ t_1\ t_2)$

Proofs automatic (more complex for *inter* and *diff*)

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Implementation of locale *Set2*:

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Correctness lemmas for *union* etc code

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Implementation of locale *Set2*:

interpretation *Set2* **where** $union = union \ldots$
and $set = set\_tree$ **and** $invar = (\lambda t.\ bst\ t \wedge inv\ t)$

`HOL/Data_Structures/Set2_Join.thy`

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join l a r — The idea

Assume l is "smaller" than r :

join l a r — The idea

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- Descend along the left spine of r until you find a subtree t of the same "size" as l:

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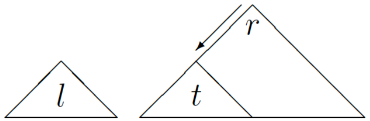

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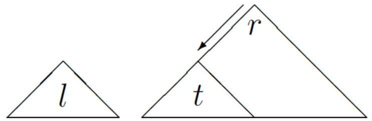

- Replace t by $\langle l, a, t \rangle$.

$join\ l\ x\ r =$
(if $bheight\ r < bheight\ l$
then $paint\ Black\ (joinR\ l\ x\ r)$
else if $bheight\ l < bheight\ r$
then $paint\ Black\ (joinL\ l\ x\ r)$ else $B\ l\ x\ r)$

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Need to store black height in each node
for logarithmic complexity

HOL/Data_Structures/
Set2_Join_RBT.thy

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Literature

The idea of "just $join$":

Stephen Adams. *Efficient Sets — A Balancing Act*.
J. Functional Programming, volume 3, number 4, 1993.

The precise analysis:

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