Games in Strategic Form & Nash Equilibrium

- New example:
  - Player 1: M not dominated by U and M not dominated by D
  - But: If Player 1 plays \( \sigma_1 = \{1/2, 0, 1/2\} \) he will get \( u(\sigma_1) = 1/2 \) regardless how player 2 plays
  - \( \Rightarrow \) a pure strategy may be dominated by a mixed strategy even if it is not strictly dominated by any pure strategy

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More Notation:

- Discussing player i’s strategy-options, holding other player’s options fixed:
  - \( s_j \in S_i; \) “other player’s strategies”
  - Short notation: \( (s_1, s_2, \ldots, s_i, s_{i+1}, \ldots, s_{n}) \)
  - Same for mixed strategies: \( (\sigma_1, \sigma_2, \ldots, \sigma_{i-1}, \sigma_i, \sigma_{i+1}, \ldots, \sigma_n) \)

Definition:

- Pure strategy \( s_i \) is strictly dominated for player i if \( \sigma'_i \) exists so that \( u(\sigma'_i, s_j) > u(s_i, s_j) \) for all \( s_j \in S_i \)
  - ... weakly dominated:
    - \( u(\sigma'_i, s_j) \geq u(s_i, s_j) \) for all \( s_j \in S_i \) (and > for at least one \( s_j \))
  - If \( u(\sigma'_i, s_j) > u(s_i, s_j) \) for all \( s_j \in S_i \) we also have \( u(\sigma'_i, \sigma''_j) > u(s_i, \sigma''_j) \) for all \( \sigma''_j \in S_i \) because \( u(\sigma'_i, \sigma_{i+1} \ldots) \) is a convex function of \( u(\sigma'_i, s_{i+1}), u(\sigma'_i, s'_{i+1}), u(\sigma'_i, s''_{i+1}), \ldots \).
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- Discussing player i’s strategy-options, holding other player’s options fixed:
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- Short notation: $(s'_i, s_i) = (s_1, ..., s_i - 1, s'_i, s_{i+1}, ..., s_n)$

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Games in Strategic Form & Nash Equilibrium

More Notation:

- Discussing player i’s strategy-options, holding other player’s options fixed:
  - $s_i \in S_i$ 
  - Short notation:
  - Same for function:
  - Strictly Convex function:
    \[ f(tx+(1-t)y) < t f(x) + (1-t)f(y) \]

Definition:

- Pure strategy
  - $u_i(\sigma'_i, s_i)$
  - ... weakly
  - $u_i(\sigma'_i, s_i) \geq u_i(s_i, s_i)$ for all $s_i \in S_i$ (and > for at least one $s_i$)

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Games in Strategic Form & Nash Equilibrium

- What about dominated mixed strategies?
  - Easy: A mixed strategy that assigns positive probabilities to pure strategies that are dominated is dominated
  - But: A mixed strategy may be dominated even if it assigns positive probabilities to pure strategies that are not even weakly dominated:

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Example:

- $U$ and $M$ are not dominated by $D$ for player 1
- But: Playing $\sigma_1 = (\frac{1}{3}, \frac{1}{3}, 0)$ gives expected utility $u_1(\sigma_1, s) = \frac{1}{2}$ no matter what 2 plays
- $D (\sigma_0 = (0, 0, 1))$ dominates $\sigma_1$

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Example:
- U and M are not dominated by D for player 1
- But: Playing $\sigma_1=(\frac{1}{3}, \frac{1}{3}, 0)$ gives expected utility $u_1(\sigma, *)=-1/2$ no matter what 2 plays $\Rightarrow D(\sigma_D=(0, 0, 1))$ dominates $\sigma_1$

A note on rationality

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- Iterated strict dominance $\Rightarrow$ (U,L)
- BUT: psychology $\Rightarrow$ play D instead of U because „U is unsafe“

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- Iterated strict dominance → (U,L)
- **BUT:** psychology → play R instead of U because "U is unsafe"

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Game Theory ↔ Decision Theory

**Example**

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- Iterated strict dominance → (U,L)
- **If player 1 reduces his payoff for U by 2:**
  - **decision theory:** no use
  - **game theory:** new iterated strict dominance → (D,R)
Games in Strategic Form & Nash Equilibrium

Game Theory ↔ Decision Theory

Example

- Iterated strict dominance \(\rightarrow (U,L)\)

\[
\begin{array}{c|cc}
 & L & R \\
\hline
U & 1, 3 & 4, 1 \\
D & 0, 2 & 3, 4 \\
\end{array}
\]

- If player 1 reduces his payoff for U by 2:
  - decision theory: no use
  - game theory: new iterated strict dominance \(\rightarrow (D,R)\)

\[
\begin{array}{c|cc}
 & L & R \\
\hline
U & -1, 3 & 2, 1 \\
D & 0, 2 & 3, 4 \\
\end{array}
\]

Games in Strategic Form & Nash Equilibrium

Prisoner’s dilemma & Iterated dominance

\[
\begin{array}{c|cc}
 & C & D \\
\hline
C & 1, 1 & -1, 2 \\
D & 2, -1 & 0, 0 \\
\end{array}
\]

- Iterated strict dominance \(\rightarrow (D,D)\)
Games in Strategic Form & Nash Equilibrium

Vickrey Auction & Iterated dominance

- **Good’s valuations:** $v_i$ ; Assume common knowledge for the moment
- **Bids:** $s_i$
- **Second price:**
  - winning condition: $s_i > \max_{i \neq j} s_j$
  - let $r_i := \max_{i \neq j} s_j$; $r_i$ is the price having to be paid
  - winner $i$’s utility: $u_i = v_i - r_i$; other players utility = 0
- for each player bidding true valuation is weakly dominant:
  - case $s_i > v_i$ : (overbidding)
    - If $r_i > s_i$ : loosely $u_i = 0$ $\rightarrow$ could have bid $v_i$ as well
    - If $r_i \leq v_i$ : wins $u_i = v_i - r_i$ $\rightarrow$ could have bid $v_i$ as well

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Games in Strategic Form & Nash Equilibrium

**Vickrey Auction & Iterated dominance**

  - case $v_i < r_i < s_i$
    - $i$ wins $u_i = v_i - r_i < 0$ (winner's curse)
      - should have bid $v_i = r_i$; $u_i = 0$ at least
  - case $s_i < v_i$ : (underbidding)
    - If $r_i \leq s_i$ or $r_i \geq v_i$
      - $u_i$ is unchanged if he bids $v_i$ instead of $s_i$
    - If $s_i \leq r_i < v_i$
      - bidder forgoes positive winning chances by underbidding

  - Assumption of common knowledge my be dropped because bidding own valuation is weakly dominant for each player
Vickrey Auction & Iterated dominance

- case $v_i < r_i < s_i$:
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Assumption of common knowledge may be dropped because bidding own valuation is weakly dominant for each player.

Nash Equilibrium

- Nash Equilibrium: strategy profile: each player’s strategy is optimal response to all other player’s strategies:

- Mixed strategy profile $\sigma^*$ is Nash Equilibrium if
  for all $i$: $u_i(\sigma^*_{-i}, \sigma^*_i) \geq u_i(S_i, \sigma^*_i)$ for all $s_i \in S_i$
  (Pure strategy profiles also possible $\Rightarrow$ “pure strategy NE”)

- Strategy profile $s^*$ is Strict Nash Equilibrium: if it is a NE and
  for all $i$: $u_i(s^*_i, s^*_{-i}) > u_i(S_i, s^*_i)$ for all $s_i \neq s^*$.
  Strict NE is necessarily a pure strategy NE by definition.
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- Nash Equilibrium: strategy profile: each player’s strategy is optimal response to all other player’s strategies.

- Mixed strategy profile $\sigma^*$ is Nash Equilibrium if for all $i$: $u_i(\sigma^*_i, \sigma^*_{-i}) \geq u_i(s_i, \sigma^*_{-i})$ for all $s_i \in S_i$ (Pure strategy profiles also possible → „pure strategy NE“)

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Nash Equilibrium

- From previous slide: $\sigma^*$ is Nash Equilibrium if for all $i$: $u_i(\sigma^*_i, \sigma^*_{-i}) \geq u_i(s_i, \sigma^*_{-i})$ for all $s_i \in S_i$

- Expected utilities are „linear in the probabilities“
  - In NE def we must only check for pure alternatives $s_i$
  - In a (non-degenerate) mixed strategy Nash Equilibrium a player must be (a priori) indifferent between all pure strategies to which he assigns positive probability (Indifference condition)

(we will analyze this in more depth later)
Indifference condition: more detailed explanation:

For player $i$'s utility, we have:

$$ u_i(\sigma) = \sum_{s_i \in S_u} \sigma_i(s_i) u_i(s_i, \sigma_{-i}) \quad \text{with} \quad \sum_{s_i \in S_u} \sigma_i(s_i) = 1 $$

for the NE $\sigma^*$ we thus have:

$$ u_i(\sigma^*) = \sum_{s_i \in S_u} \sigma_i^*(s_i) u_i(s_i, \sigma_{-i}^*) \quad \text{with} \quad \sum_{s_i \in S_u} \sigma_i^*(s_i) = 1 $$

since $u_i(\sigma^*)$ is the best outcome, I can achieve, when the others play $\sigma_{-i}^*$, all the $u_i(s_i, \sigma_{-i}^*)$ with $\sigma_i(s_i) > 0$ must be equal, and equal to $u_i(\sigma^*)$.

why? $\rightarrow$ no $u_i(s_i, \sigma_{-i}^*)$ can be greater than $u_i(\sigma^*)$ otherwise the NE condition would be violated, and also not smaller, because then the sum would also be smaller.

Indifference condition: more detailed explanation:

For player $i$'s utility, we have:

$$ u_i(\sigma) = \sum_{s_i \in S_u} \sigma_i(s_i) u_i(s_i, \sigma_{-i}) \quad \text{with} \quad \sum_{s_i \in S_u} \sigma_i(s_i) = 1 $$

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$$ u_i(\sigma^*) = \sum_{s_i \in S_u} \sigma_i^*(s_i) u_i(s_i, \sigma_{-i}^*) \quad \text{with} \quad \sum_{s_i \in S_u} \sigma_i^*(s_i) = 1 $$

since $u_i(\sigma^*)$ is the best outcome, I can achieve, when the others play $\sigma_{-i}^*$, all the $u_i(s_i, \sigma_{-i}^*)$ with $\sigma_i(s_i) > 0$ must be equal, and equal to $u_i(\sigma^*)$.

why? $\rightarrow$ no $u_i(s_i, \sigma_{-i}^*)$ can be greater than $u_i(\sigma^*)$ otherwise the NE condition would be violated, and also not smaller, because then the sum would also be smaller.
Nash Equilibrium

- From previous slide: $\sigma^*$ is Nash Equilibrium if for all $i$: $u_i(\sigma^{*,i}, \sigma^{*,\cdot}) \geq u_i(s_i, \sigma^{*,\cdot})$ for all $s_i \in S_i$
- Expected utilities are "linear in the probabilities"

$\rightarrow$ In NE def we must only check for pure alternatives $s_i$

$\rightarrow$ In a (non-degenerate) mixed strategy Nash Equilibrium a player must be (a priori) indifferent between all pure strategies to which he assigns positive probability (Indifference condition)

(we will analyze this in more depth later)

Indifference condition: more detailed explanation:

For player i's utility, we have:

$$u_i(\sigma) = \sum_{s_i \in S_u} \sigma_i(s_i) u_i(s_i, \sigma_{-i}) \quad \text{with} \quad \sum_{s_i \in S_u} \sigma_i(s_i) = 1$$

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why? $\rightarrow$ no $u_i(s_i, \sigma_{-i}^*)$ can be greater than $u_i(\sigma^*)$ otherwise the NE condition would be violated, and also not smaller, because then the sum would also be smaller.
Nash Equilibrium

- Strict equilibria need not exist. However each finite strategy form game has a mixed strategy equilibrium.
- In NE no player has incentive to deviate from NE
- In reality: If rationality is "non-strict" (mistakes are made): deviations can occur
- If one round of elimination of strictly dominated strategies yields unique strategy profile, this strategy profile is a strict NE (unique)
- In NE, positive probabilities may only be assigned to not-strictly dominated strategies (Otherwise profit may be increased by choosing a dominating strategy).

Nash Equilibrium: Example: Cournot Competition

- Cournot model: Duopoly. Each of two firms (players) i produces same good.
- Output levels \( q_i \) are chosen from sets \( Q_i \)
- Cost of production is \( c_i(q_i) \)
- Market price is \( p(q) = p(q_1 + q_2) \)
- Firm i's profit is then \( u_i(q_i, q_j) = q_i p(q) - c_i(q_i) \)
- Cournot reaction functions \( r_1 : Q_2 \rightarrow Q_1 \) and \( r_2 : Q_1 \rightarrow Q_2 \) specify optimal reaction on output level of opponent
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Under certain reasonable assumptions (see [1]) we can maximize e.g. \( u_2(q_1, q_2) \) by solving \( \frac{du_2}{dq_2} = u_2(q_1, q_2) = p(q_1, q_2) + p'(q_1, q_2) q_2 - c_2(q_1) = 0 \).

Inserting \( r_2(q_1) \) for \( q_2 \)

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p(q_1 + r_2(q_1)) + p'(q_1 + r_2(q_1)) r_2(q_1) - c_2'(r_2(q_1)) = 0
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gives the defining equation for \( r_2(.) \).

(analogous for \( r_1(.) \)).

The intersections of the functions \( r_2 \) and \( r_1 \) are the NE of the Cournot game.

**Example:** Linear demand \( p(q) = \max(0, 1-q) \); linear cost: \( c_i(q_i) = c q_i \);

\[
\rightarrow r_2(q_1) = 1/2 (1 - q_1 - c); \quad r_1(q_2) = 1/2 (1 - q_2 - c);
\]

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Nash Equilibrium: Example: Cournot Competition

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Games in Strategic Form & Nash Equilibrium

Nash Equilibrium: Example: Cournot Competition

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Games in Strategic Form & Nash Equilibrium

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Games in Strategic Form & Nash Equilibrium

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Games in Strategic Form & Nash Equilibrium

Nash Equilibrium: Example: Hotelling Competition

- Two firms, 1 (at x=0) and 2 (at x=1) sell same good
- Unit cost of product := c; price for product of firm i := \( p_i \)
- Customers: uniformly distributed over [0,1] with probability density 1
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- Customers: have unit demand;
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Games in Strategic Form & Nash Equilibrium

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Nash Equilibrium: Example: Hotelling Competition

- Demand for firm 1 is $D_1(p_1, p_2) = x \text{ where } p_1 + tx = p_2 + t(1-x)$
- \( D_1(p_1, p_2) = (p_2 - p_1) + t \) \( / (2t) \)
- \( D_1(p_1, p_2) = 1 - D_2(p_1, p_2) \)
- Nash Equilibrium \((p^*_1, p^*_2)\): For each i: \( p^*_i \in \text{argmax} \{ (p_i - c) D_i(p_i, p^*_i) \} \)
- Denoting the reaction functions by \( r_1(p_2) \) and \( r_2(p_1) \) we get for e.g. firm 2:
  \[
  \frac{d}{dp_1} \left\{ (p_2 - c) D_2(p^*_1, p_2) \right\} = 0
  \]
  Afterwards insert \( r_2(p_1) \) for \( p_2 \) \( \rightarrow \)
  \[
  D_2(p_1, r_2(p_1)) + (r_2(p_1) - c) \frac{d}{dp_2} D_2(p_1, r_2(p_1)) = 0
  \]
  \[
  p^*_1 = p^*_2 = c + t \text{ for } c + 3/2 \leq \bar{s}
  \]

---

Games in Strategic Form & Nash Equilibrium

Nash Equilibrium: Example: Hotelling Competition

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\[
\begin{align*}
  & x=0 & \quad x=1 \\
  & \quad 1 & \quad 2
\end{align*}
\]
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  D_2(p_1, r_2(p_1)) + (r_2(p_1) - c) d/dp_2 D_2(p_1, r_2(p_1)) = 0 \quad \text{for } c + 3/2t \leq 5
  \]

Nash Equilibrium: Non-Existence-of Pure NE-Example

- Some games may have more than one pure strategy NE
- Not all games have a pure strategy NE:
- Example: Matching pennies:
  \[
  \begin{array}{c|c|c}
  & H & T \\ \hline
  H & 1, -1 & -1, 1 \\ T & -1, 1 & 1, -1
  \end{array}
  \]
- Both players simultaneously announce
- Head or Tails: IF match $\rightarrow$ 1 wins; If differ $\rightarrow$ 2 wins
- No pure NE;
  but mixed strategy NE: $(1/2, 1/2); (1/2, 1/2)$:
- Reasoning: If player 2 plays (1/2, 1/2) then player 1’s expected payoff is $1/2 * 1 + 1/2 * (-1) = 0$ when playing $H$ and $1/2 * (-1) + 1/2 * 1 = 0$ when playing $T$. Player 1 is also indifferent.

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Reasoning: If player 2 plays (1/2, 1/2) then player 1’s expected payoff is 1/2 * 1 + 1/2 * (-1) = 0 when playing H and 1/2 * 1 + 1/2 * (-1) = 0 when playing T → player 1 is also indifferent

Another example: Inspection game
- Worker: work or shirk; Employer: Inspect or not inspect
- Worker: working costs $g$, produces value $v$; gets wage $w$
- Employer: Inspection costs $h$
- We assume $w > g > h > 0$
- If not inspect → worker shirks → better inspect → if inspect → worker always works → better not inspect → ....: No pure NE
- Employer must randomize
Games in Strategic Form & Nash Equilibrium

Nash Equilibrium: Non-Existence—of Pure NE—Example 2

- Another example: Inspection game
  - Worker: work or shirk; Employer: Inspect or not inspect
  - Worker: working costs $g$, produces value $v$; gets wage $w$
  - Employer: Inspection costs $h$
  - We assume $w > g > h > 0$
  - If not inspect $\rightarrow$ worker shirks $\rightarrow$ better inspect $\rightarrow$ if inspect $\rightarrow$ worker always works $\rightarrow$ better not inspect $\rightarrow$ ...: No pure NE
  - $\rightarrow$ Employer must randomize

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<tr>
<td>$S$</td>
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Games in Strategic Form & Nash Equilibrium

Nash Equilibrium: Non-Existence—of Pure NE—Example 2

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\[
\begin{array}{c|c|c}
\text{Customer} & \text{Location} & \text{Decision} \\
\hline
1 & x=0 & \text{Buy from firm 1} \\
2 & x=1 & \text{Buy from firm 2} \\
\end{array}
\]
### Games in Strategic Form & Nash Equilibrium

#### Nash Equilibrium: More than one NE

- **Another example: Battle of the sexes**
  
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Nash Equilibrium: More than one NE

- Another example: Battle of the sexes
  - Two pure NE: (F,F) and (B,B)
  - One mixed NE: Indifference condition
    \( \sigma_1(F) = x \) and \( \sigma_2(B) = y \)
    Player 1's indifference:
    \( 0 + 2(1-y) = 1 \Rightarrow y = 2/3 \)
    Player 2's indifference:
    \( 0 + 2(1-x) = 1 \Rightarrow x = 2/3 \)
    Mixed NE: \((2/3, 1/3); (2/3, 1/3))

- Another example: Game of chicken
  - (same reasoning)
  - Mixed NE: \((1/2, 1/2); (1/2, 1/2))

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Games in Strategic Form & Nash Equilibrium

Nash Equilibrium: More than one NE

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    \( \sigma_1(F) = x \) and \( \sigma_2(B) = y \)
    Player 1's indifference:
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    Player 2's indifference:
    \( 0 + 2(1-x) = 1 \Rightarrow x = 2/3 \)
    Mixed NE: \((2/3, 1/3); (2/3, 1/3))

- Another example: Game of chicken
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Nash Equilibrium: More than one NE

- Another example: Battle of the sexes
  - Two pure NE: (F,F) and (B,B)
  - One mixed NE: Indifference condition
    \( \sigma_1(F) = x \) and \( \sigma_2(B) = y \)
    Player 1's indifference:
    \( 0 + 2(1-y) = 1 \Rightarrow y = 2/3 \)
    Player 2's indifference:
    \( 0 + 2(1-x) = 1 \Rightarrow x = 2/3 \)
    Mixed NE: \((2/3, 1/3); (2/3, 1/3))

- Another example: Game of chicken
  - (same reasoning)
  - Mixed NE: \((1/2, 1/2); (1/2, 1/2))

<table>
<thead>
<tr>
<th></th>
<th>T</th>
<th>W</th>
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<tbody>
<tr>
<td>T</td>
<td>-1, -1</td>
<td>2, 1</td>
</tr>
<tr>
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Nash Equilibrium: More than one NE

Focal points

- Some games have more than one NE \( \Rightarrow \) which will be chosen?
- Theory of „focalness“ of NE („focal points“):
  Example: Chose time of day simultaneously; reward if match: 12 noon is focal, 15:37 is not

Risk Dominance

- Stag Hunt: NE: \((C,C)\) and \((D,D)\); \((C,C)\) is pareto-dominant \( \Rightarrow \) \((C,C)\) might be chosen if \( p(C)<0.5 \) BUT
- more than two players: ALL have to agree on \( C \) \( \Rightarrow p(C)<0.5 \) \( p(C)>0.93 \) \( \Rightarrow \) \((D,D)\) „risk dominates“ \((C,C)\)

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Games in Strategic Form & Nash Equilibrium

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Nash Equilibrium: More than one NE

Risk Dominance / Pareto Optimality

- In this game: (Among others) two pure NE: (U,L) and (D,R); (U,L): Pareto dominates (D,R)
- But: For player 1 D is safer (guarantees min payoff of 7) \( \rightarrow \) if \( p(R) > 1/8 \) don’t go for (U,L) \( \rightarrow \) no certainty!
- Pregame-communication / agreement on (U,L) ?!
  No: player 2 gains if player 1 plays U \( \rightarrow \) player 2 will always tell „L” regardless of true intentions \( \rightarrow \) agreement is worthless

<table>
<thead>
<tr>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>9, 9</td>
</tr>
<tr>
<td>D</td>
<td>8, 0</td>
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</table>

Nash Equilibrium: More than one NE

Risk Dominance / Pareto Optimality

- Three player game: Two pure NE: (U,L,A) and (D,R,B); (and one mixed); (U,L,A) pareto-dominates (D,R,B)
- If player 3’s choice is fixed \( \rightarrow \) Two player game \( \rightarrow \) (D,R) is pareto-dominant \( \rightarrow \) if players 1 and 2 expect A : coordinate on (D,R).
- \( \rightarrow \) concept of „coalition proof eq.“ (here (D,R,B))(see [1])

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<td>U</td>
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</tr>
<tr>
<td>D</td>
<td>-5,-5,0</td>
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<th>R</th>
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<tbody>
<tr>
<td>A</td>
<td>1,1,-5</td>
</tr>
<tr>
<td>B</td>
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Games in Strategic Form & Nash Equilibrium

### Nash Equilibrium: More than one NE

**Risk Dominance / Pareto Optimality**

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- If player 3’s choice is fixed → Two player game → (D,R) is pareto-dominant → if players 1 and 2 expect A: coordinate on (D,R).
- → concept of “coalition proof eq.” (here (D,R,B))(see [1])

### Games in Strategic Form & Nash Equilibrium

**Mixed Nash Equilibrium: General Analysis for 2 x 2 Games**

(see [2])

<table>
<thead>
<tr>
<th>Pure NE: One cell ➔</th>
<th>Player B</th>
<th>q</th>
<th>1-q</th>
</tr>
</thead>
<tbody>
<tr>
<td>For A: cell’s payoff for A must be (weak) maximum over rows in that column</td>
<td>Player A</td>
<td>p</td>
<td>1-p</td>
</tr>
<tr>
<td>For B: cell’s payoff for B must be (weak) maximum over column in that row</td>
<td></td>
<td>a_{UL}, b_{UL}</td>
<td>a_{UR}, b_{UR}</td>
</tr>
</tbody>
</table>

- Example: (U,R) is pure NE if \( a_{UR} \geq a_{DR} \) and \( b_{UR} \geq b_{UL} \)