Basics

- Basic scenario: Players simultaneously choose action to perform $\rightarrow$ result of the actions they select $\rightarrow$ outcome in discrete state space $\Omega$
- outcome depends on the combination of actions
- Assume: each player has just two possible actions $C$ ("cooperate") and $D$ ("defect")
- Environment behavior given by state transformer function:
  $$ \tau : \mathcal{A} \times \mathcal{A} \rightarrow \Omega $$
  Player $i$'s action $\times$ Player $j$'s action

Rational Behavior

- Assumption: Environment is sensitive to actions of both players:
  $$ \tau(D, D) = \omega_1 \quad \tau(D, C) = \omega_2 \quad \tau(C, D) = \omega_3 \quad \tau(C, C) = \omega_4 $$
- Utility functions:
  $$ u_i(\omega_1) = 1 \quad u_i(\omega_2) = 1 \quad u_i(\omega_3) = 4 \quad u_i(\omega_4) = 4 $$
  $$ u_j(\omega_1) = 1 \quad u_j(\omega_2) = 4 \quad u_j(\omega_3) = 1 \quad u_j(\omega_4) = 4 $$
- Short notation:
  $$ u_i(D, D) = 1 \quad u_i(D, C) = 1 \quad u_i(C, D) = 4 \quad u_i(C, C) = 4 $$
  $$ u_j(D, D) = 1 \quad u_j(D, C) = 4 \quad u_j(C, D) = 1 \quad u_j(C, C) = 4 $$
- $\rightarrow$ player's preferences:
  (also in short notation): $C, C \succ_i C, D \succ_i D, C \succ_i D, D$
**Assumption:** Environment is sensitive to actions of both players: \( \tau(D, D) = \omega_1 \), \( \tau(D, C) = \omega_2 \), \( \tau(C, D) = \omega_3 \), \( \tau(C, C) = \omega_4 \).

**Utility functions:**
- \( u_i(\omega_1) = 1 \)_i
- \( u_i(\omega_2) = 1 \)_i
- \( u_i(\omega_3) = 4 \)_i
- \( u_i(\omega_4) = 4 \)_i

\( u_j(\omega_1) = 1 \)_j
\( u_j(\omega_2) = 4 \)_j
\( u_j(\omega_3) = 1 \)_j
\( u_j(\omega_4) = 4 \)_j

**Short notation:**
- \( u_i(D, D) = 1 \)_i
- \( u_i(D, C) = 1 \)_i
- \( u_i(C, D) = 4 \)_i
- \( u_i(C, C) = 4 \)_i

\( u_j(D, D) = 1 \)_j
\( u_j(D, C) = 4 \)_j
\( u_j(C, D) = 1 \)_j
\( u_j(C, C) = 4 \)_j

→ player's preferences:
(also in short notation):
- \( C, C \succ_i C, D \)
- \( C, D \succ_i D, D \)

**Game theory:** characterize the previous scenario in a payoff matrix:

\[
\begin{array}{cc|cc}
  & \text{defect} & \text{coop} \\
\hline
\text{defect} & 1 & 4 \\
\text{coop} & 1 & 4 \\
\end{array}
\]

\( C \) is the **rational choice** for \( i \).
(Because \( i \) (strongly) prefers all outcomes that arise through \( C \) over all outcomes that arise through \( D \).)

\( C \) is the **rational choice** for \( j \).
(Because \( j \) (strongly) prefers all outcomes that arise through \( C \) over all outcomes that arise through \( D \).)

**Player** \( i \) is “column player”
**Player** \( j \) is “row player”
Rational Behavior

• Game theory: characterize the previous scenario in a payoff matrix:

<table>
<thead>
<tr>
<th></th>
<th>defect</th>
<th>coop</th>
</tr>
</thead>
<tbody>
<tr>
<td>defect</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>coop</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

same as:  

\[
\begin{align*}
& u_i(D, D) = 1 & u_i(D, C) = 1 & u_i(C, D) = 4 & u_i(C, C) = 4 \\
& u_j(D, D) = 1 & u_j(D, C) = 4 & u_j(C, D) = 1 & u_j(C, C) = 4
\end{align*}
\]

• Player i is “column player”
• Player j is “row player”

Rational Behavior

\[
\begin{align*}
& u_i(D, D) = 1 & u_i(D, C) = 1 & u_i(C, D) = 4 & u_i(C, C) = 4 \\
& u_j(D, D) = 1 & u_j(D, C) = 4 & u_j(C, D) = 1 & u_j(C, C) = 4
\end{align*}
\]

\[
\begin{align*}
& C, C \succeq_i C, D \succ_i D, C \succeq_i D, D \\
& C, C \succeq_j D, C \succ_j C, D \succeq_j D, D
\end{align*}
\]

• “C” is the rational choice for i.
  (Because i (strongly) prefers all outcomes that arise through C over all outcomes that arise through D.)

• “C” is the rational choice for j.
  (Because j (strongly) prefers all outcomes that arise through C over all outcomes that arise through D.)

Dominant Strategies and Nash Equilibria

• With respect to “what should I do”:
If \( \Omega = \Omega_1 \cup \Omega_2 \) we say \( \Omega_1 \) weakly dominates \( \Omega_2 \) for player i “iff for player i every state (outcome) in \( \Omega_1 \) is preferable to or at least as good as every state in \( \Omega_2 \):

\[
\forall \omega_1 \forall \omega_2 : (\omega_1 \in \Omega_1 \land \omega_2 \in \Omega_2) \rightarrow \omega_1 \succeq \omega_2
\]

• If \( \Omega = \Omega_1 \cup \Omega_2 \) we say \( \Omega_1 \) strongly dominates \( \Omega_2 \) for player i “iff for player i every state (outcome) in \( \Omega_1 \) is preferable to every state in \( \Omega_2 \):

\[
\forall \omega_1 \forall \omega_2 : (\omega_1 \in \Omega_1 \land \omega_2 \in \Omega_2) \rightarrow \omega_1 \succ \omega_2
\]

• Example:
\[
\begin{align*}
\Omega_1 & = \{ \omega_1, \omega_2, \omega_3, \omega_4 \} & \Omega_2 & = \{ \omega_3, \omega_4 \} \\
\omega_1 & \succ_i \omega_2 & \omega_3 & \succ_i \omega_4
\end{align*}
\]

\( \Omega_1 \) strongly dominates \( \Omega_2 \) for player i“.
Dominant Strategies and Nash Equilibria

- With respect to "what should I do":
  If \( \Omega = \Omega_1 \cup \Omega_2 \) we say \( \Omega_1 \) weakly dominates \( \Omega_2 \) for player \( i \) iff for player \( i \) every state (outcome) in \( \Omega_1 \) is preferable to or at least as good as every state in \( \Omega_2 \):
  \[
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  \]

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  \]

- Example:
  \[
  \Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\} \quad \Omega_1 = \{\omega_1, \omega_2\}
  \quad \Omega_2 = \{\omega_3, \omega_4\}
  \]
  \( \Omega_1 \) strongly dominates \( \Omega_2 \) for player \( i \):

Rational Behavior

- Game theory notation: actions are called "strategies"
- Notation: \( s^* \) is the set of possible outcomes (states) when "playing strategy \( s \)" (executing action \( s \))
- Example: if we have (as before):
  \[
  \tau(D, D) = \omega_1 \quad \tau(D, C) = \omega_2 \quad \tau(C, D) = \omega_3 \quad \tau(C, C) = \omega_4
  \]
  we have (from player \( i \)'s point of view):
  \[
  D^* = \{\omega_1, \omega_2\} \quad C^* = \{\omega_3, \omega_4\}
  \]
- Notation: "strategy \( s_1 \) (strongly / weakly) dominates \( s_2 \)" iff \( s_1^* \) (strongly / weakly) dominates \( s_2^* \)
- If one strategy strongly dominates the other \( \rightarrow \) question what to do is easy. (do first)

Competitive and Zero-Sum Interactions

- Scenario ("strictly competitive"): Player \( i \) prefers outcome \( \omega \) over \( \omega' \) iff player \( j \) prefers outcome \( \omega \) over \( \omega' \):
  \[
  \omega \succeq_i \omega' \iff \omega' \succeq_j \omega
  \]

- Scenario ("zero-sum"): \( \forall \omega \in \Omega : u_i(\omega) + u_j(\omega) = 0 \)

- Zero-sum games are always strictly competitive
- Zero-sum games imply negative utility for "loser"
- Strictly zero-sum: only in games like chess. Real world never "strictly zero-sum" (Example: two girls compete to win the heart of the same guy). But: Unfortunately many encounters are perceived as zero sum games.
Competitive and Zero-Sum Interactions

- Scenario ("strictly competitive"): Player $i$ prefers outcome $\omega$ over $\omega'$ iff player $j$ prefers outcome $\omega'$ over $\omega$:
  \[ \omega \succ_i \omega' \iff \omega' \succ_j \omega \]

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- Zero-sum games are always strictly competitive

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- Strictly zero-sum: only in games like chess. Real world never "strictly zero-sum" (Example: two girls compete to win the heart of the same guy). But: Unfortunately many encounters are perceived as zero sum games.

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The Prisoner’s Dilemma

- Two criminals are held in separate cells (no communication):
  1. One confesses and the other does not \rightarrow confessor is freed and the other gets 3 years
  2. Both confess \rightarrow each gets 2 years
  3. Neither confesses \rightarrow both get 1 year

- Associations: Confess == D; Not Confess == C

- Payoff matrix

<table>
<thead>
<tr>
<th></th>
<th>i defects</th>
<th>i cooperates</th>
</tr>
</thead>
<tbody>
<tr>
<td>j defects</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>j cooperates</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>

- Take place of prisoner (e.g. prisoner $i$) \rightarrow 

  Course of Reasoning:

  - Suppose I cooperate: If $j$ also cooperates \rightarrow we both get payoff 3. If $j$ defects \rightarrow I get payoff 0. \rightarrow Best guaranteed payoff when I cooperate is 0
  - Suppose I defect: If $j$ cooperates \rightarrow I get payoff 5. If $j$ also defects \rightarrow both get payoff 2. \rightarrow Best guaranteed payoff when I defect is 2
  - \rightarrow If I defect I’ll get a minimum guaranteed payoff of 2. If I cooperate I’ll get a minimum guaranteed payoff of 0.
  - \rightarrow If prefer guaranteed payoff of 2 to guaranteed payoff of 0.
    \rightarrow I should defect
The Prisoner's Dilemma

\[\begin{align*}
\text{iD} & \quad \text{iC} \\
\text{jD} & \quad 2 & 2 & 5 & 0 \\
\text{jC} & \quad 0 & 5 & 3 & 3 \\
\end{align*}\]

\[\begin{align*}
\text{u}_i(D,D) = 2, & \quad \text{u}_i(D,C) = 5, & \quad \text{u}_i(C,D) = 0, & \quad \text{u}_i(C,C) = 3 \\
\text{u}_j(D,D) = 2, & \quad \text{u}_j(D,C) = 0, & \quad \text{u}_j(C,D) = 5, & \quad \text{u}_j(C,C) = 3 \\
\end{align*}\]

\[\begin{align*}
(D,C) & \succ_j (C,C) \succ_j (D,D) \succ_j (C,D) \\
(C,D) & \succ_j (C,C) \succ_j (D,D) \succ_j (D,C) \\
\end{align*}\]

**Take place of prisoner (e.g. prisoner i) → Course of Reasoning:**

- Suppose I cooperate: If \( j \) also cooperates → we both get payoff 3. If \( j \) defects → I get payoff 0. ⇒ Best guaranteed payoff when I cooperate is 0
- Suppose I defect: If \( j \) cooperates → I get payoff 5. If \( j \) also defects → both get payoff 2. ⇒ Best guaranteed payoff when I defect is 2
  - If I defect I’ll get a minimum guaranteed payoff of 2. If I cooperate I’ll get a minimum guaranteed payoff of 0.
  - If prefer guaranteed payoff of 2 to guaranteed payoff of 0. ⇒ I should defect

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  - If prefer guaranteed payoff of 2 to guaranteed payoff of 0. ⇒ I should defect
The Prisoner’s Dilemma

- only one Nash equilibrium: \((D,D)\). (under the assumption that the other does D, one can do no better than do D)
- Intuition says: \((C,C)\) is better than \((D,D)\) so why not \((C,C)\)? 
  \(\rightarrow\) but if player assumes that other player does C it is BEST to do D! \(\rightarrow\) seemingly „waste of utility“
- „shocking“ truth: defection is rational, cooperate is irrational
- Other prisoner’s dilemma: Nuclear arms reduction (D: do not reduce, C: reduce)

The shadow of the future: Iterated Prisoner’s Dilemma Game

- Game is played multiple times. Players can see all past actions of other player.
- Course of reasoning:
  - If I defect, the other player may punish me by defecting in the next run. (not a point in the one shot Prisoner’s Dilemma game)
  - Testing cooperation (and possibly getting the sucker’s payoff) is not tragic, because „on the long run“ one (or several) sucker’s payoff[s] is (are) „statistically“ not important (can e.g. be equaled by gains through mutual cooperation)
- \(\rightarrow\) in an iterated PD-game: cooperation is rational

- „Defect more rational than cooperate“ \(\rightarrow\) Humans: Machiavellism (opposed to real altruism)
- Philosophical question: isn’t even altruism ultimately some kind of optimization towards OWN goals?!
- Further aspect: Strict rationalism (in case of prisoner’s dilemma: defect) is usually only applied when sucker’s payoff really hurts.
- What we have not yet regarded: Multiple sequential games between same players \(\rightarrow\) „The shadow of the future“ \(\rightarrow\) What does it mean for rationalism and strategy?
Competing PD-strategies: Axelrod’s tournament (1980)

(1) Do not be envious: Not necessary to „beat“ opponent to do well

(2) Do not be first to defect: Cooperation is risky (sucker’s payoff) but overall, some losses do not count that much and cooperation may result in win-win-situations (C,C)

(3) Reciprocate C and D: TIT-FOR-TAT balances punishing and forgiving → encourages cooperation for other player. TIT-FOR-TAT is fair: retaliates exactly with the same amount of maliciousness as opponent

(4) Don't be too clever: TIT-FOR-TAT was simplest but won over programs with complex models of opponent’s strategies:

Other symmetric 2x2 Games

* „2x2“: two players, each with two actions; Symmetric:

\[(D,C) \succ_i (C,C) \succ_j (D,D) \succ_j (C,D)\]

\[(C,D) \succ_i (C,C) \succ_j (D,D) \succ_j (C,D)\]

• Other symmetric 2x2 games (There are 4!=24 such games):

<table>
<thead>
<tr>
<th>(D,C)</th>
<th>(C,C)</th>
<th>(D,D)</th>
<th>(C,D)</th>
</tr>
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<tr>
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</table>

Prisoner’s Dilemma
Game of Chicken
Stag Hunt
Defection dominates
Defection dominates
Cooperation dominates
Cooperation dominates

• Two Nash equilibria: (D,D), (C,C)

(Stag Hunt)

Going back to J.J. Rousseau (1775)
Modern variant: You and a friend decide: good joke to appear both naked on a party. C: really do it; D: not do it

\[(C,C) \succ_i (D,C) \succ_j (D,D) \succ_j (C,D)\]

j:D 1 2 0
j:C 0 2 3

(Assuming the other does D you can do no better than do D Assuming the other does C you can do no better than do C)
Other symmetric 2x2 Games

Stag Hunt

- Going back to J.J. Rousseau (1775)

- Modern variant: You and a friend decide: good joke to appear both naked on a party. \( C: \) really do it; \( D: \) not do it

\[
(C, C) \succ_i (D, C) \succ_i (D, D) \succ_i (C, D)
\]

\[
\begin{array}{c|c|c}
j & D & C \\
\hline
i & 1 & 2 \\
\hline
C & 0 & 3 \\
\hline
\end{array}
\]

- Two Nash equilibria: \( (D, D) \), \( (C, C) \)
  (Assuming the other does \( D \) you can do no better than do \( D \)
  Assuming the other does \( C \) you can do no better than do \( C \))

Game of Chicken

- Going back to a James Dean film

- Modern variant: Gangster and hero drive cars directly towards each other \( C: \) steer away; \( D: \) not steer away

\[
(D, C) \succ_i (C, C) \succ_i (C, D) \succ_i (D, D)
\]

\[
\begin{array}{c|c|c}
j & D & C \\
\hline
i & 0 & 1 \\
\hline
C & 3 & 2 \\
\hline
\end{array}
\]

- Two Nash equilibria: \( (D, C) \), \( (C, D) \)
  (Assuming the other does \( D \) you can do no better than do \( C \)
  Assuming the other does \( C \) you can do no better than do \( D \))
Other symmetric 2x2 Games

Game of Chicken

- Going back to a James Dean film
- Modern variant: Gangster and hero drive cars directly towards each other C: steer away; D: not steer away

\[(D,C)\succ_i (C,C)\succ_i (C,D)\succ_i (D,D)\]

- Two Nash equilibria: (D,C), (C,D)
  (Assuming the other does D you can do no better than do C
   Assuming the other does C you can do no better than do D)

Notation: Strategic Form Games

- Set $\delta$ of players: \{1,2,..,1\}
  Example: \{1,2\}

- Player index: $i \in \delta$

- Pure Strategy Space $S$ of player i
  Example: $S_i = \{U,M,D\}$ and $S_i = \{L,M,R\}$

- Strategy profile $s = (s_1,..,s_i)$ where each $s_i \in S_i$
  Example: $(D, M)$

- (Finite) space $S = \chi, S_i$ of strategy profiles $s \in S$
  Example: $S = \{(U,L), (U,M), ..., (D,R)\}$

- Payoff function $u_i: S \rightarrow \mathbb{R}$ gives von Neumann-Morgenstern-utility $u_i(s)$
  for player i of strategy profile $s \in S$
  Examples: $u_i((U,L)) = 4$, $u_i((U,U)) = 3$, $u_i((M,M)) = 8$ ....

- Set of player $i$'s opponents: "-i"
  Example: $-i = \{2\}$
Notation: Strategic Form Games

- Set \( \delta \) of players: \( \{1,2,\ldots, l\} \)
  
  **Example:** \( \{1,2\} \)

- Player index: \( i \in \delta \)

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>M</th>
<th>R</th>
</tr>
</thead>
</table>
  U  | 4,3| 5,1| 6,2|
  M  | 2,1| 8,4| 3,6|
  D  | 3,0| 9,6| 2,8|

- Pure Strategy Space \( S_i \) of player \( i \)
  
  **Example:** \( S_i = \{U,M,D\} \) and \( S_j = \{L,M,R\} \)

- Strategy profile \( s = (s_1, \ldots, s_l) \) where each \( s_i \in S_i \)
  
  **Example:** \( (D,M) \)

- (Finite) space \( S = \times_i S_i \) of strategy profiles \( s \in S \)
  
  **Example:** \( S = \{ (U,L), (U,M), \ldots, (D,R) \} \)

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  **Examples:** \( u_i((U,L)) = 4 \), \( u_i((U,L)) = 3 \), \( u_i((M,M)) = 8 \) ......

  **Set of player \( i \)'s opponents: \( \sim i \)\)
  
  **Example:** \( i = \{2\} \)

---

Notation: Strategic Form Games

- Two Player zero sum game:
  
  \[ \forall S : \sum_{i=1}^{2} u_i(s) = 0 \]

- Structure of game is common knowledge:
  all players know;
  all players know that all players know;
  all players know that all players know that all players know;
  ....

- Mixed strategy \( \sigma_i : S_i \rightarrow [0,1] \) Probability distribution over pure strategies (statistically independent for each player);
  
  **Examples:** \( \sigma_i(U) = 1/3 \), \( \sigma_i(M) = 2/3 \), \( \sigma_i(D) = 0 \);
  
  \( \sigma_i(U) = 2/3 \), \( \sigma_i(M) = 1/6 \), \( \sigma_i(D) = 1/6 \);

  **Thus:** \( \sigma_i(s_i) \) is the probability that player \( i \) assigns to strategy (action) \( s_i \)

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<th>M</th>
<th>R</th>
</tr>
</thead>
</table>
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Two Player zero sum game:
\[ \forall S : \sum_{i=1}^{2} u_i(S) = 0 \]

Structure of game is common knowledge:
- all players know;
- all players know that all players know;
- all players know that all players know that all players know;

Mixed strategy \( \sigma_i : S_i \rightarrow [0,1] \) Probability distribution over pure strategies (statistically independent for each player);

Examples:
- \( \sigma_i(U)=1/3, \sigma_i(M)=2/3, \sigma_i(D)=0; \)
- \( \sigma'_i(U)=2/3, \sigma'_i(M)=1/6, \sigma'_i(D)=1/6; \)

Thus: \( \sigma_i(s_i) \) is the probability that player \( i \) assigns to strategy (action) \( s_i \)

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Example: Rock Paper Scissors

<table>
<thead>
<tr>
<th></th>
<th>Rock</th>
<th>Paper</th>
<th>Scissors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rock</td>
<td>0,0</td>
<td>-1,1</td>
<td>-1,1</td>
</tr>
<tr>
<td>Paper</td>
<td>-1,1</td>
<td>0,0</td>
<td>-1,1</td>
</tr>
<tr>
<td>Scissors</td>
<td>-1,1</td>
<td>-1,3</td>
<td>0,0</td>
</tr>
</tbody>
</table>

No pure NE, but mixed NE if both play \((1/3, 1/3, 1/3)\)
• Space of mixed strategies for player $i$: $\Sigma_i$
• Space of mixed strategy profiles: $\Sigma = \mathcal{X}_i \times \Sigma_i$
• Mixed strategy profile $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_i) \in \Sigma$
• Player $i$'s payoff when a mixed strategy profile $\sigma$ is played is
  $$\sum_{s \in S} \left( \prod_{j=1}^{f} \sigma_j(s_j) \right) u_i(s)$$
denoted as $u_i(\sigma)$, is a linear function of the $\sigma_i$
• A pure strategy of a player is a special mixed strategy of that player with one probability equal to 1 and all others equal to 0
### Example:

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or short

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\sigma_1 = (1/3, 1/3, 1/3) \\
\sigma_2 = (0, 1/2, 1/2)
\]

We then have:

\[
u_1(\sigma_1, \sigma_2) = \frac{1}{3} (0 \cdot 4 + \frac{1}{3} \cdot 5 + \frac{1}{3} \cdot 6) + \frac{1}{3} (0 \cdot 2 + \frac{1}{3} \cdot 8 + \frac{1}{3} \cdot 3) + \frac{1}{3} (0 \cdot 3 + \frac{1}{3} \cdot 9 + \frac{1}{3} \cdot 2) = 11/2
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u_2(\sigma_1, \sigma_2) = ... = 27/6
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Games in Strategic Form & Nash Equilibrium

- What is rational to do?
  - No matter what player 1 does: \( R \) gives player 2 a strictly higher payoff than \( M \).
    \( M \) is strictly dominated by \( R \)
  - \( U \rightarrow \) player 1 knows that player 2 will not play \( M \rightarrow U \) is better than \( M \) or \( D \)
  - \( M \rightarrow \) player 2 knows that player 1 knows that player 2 will not play \( M \rightarrow \) player 2 knows that player 1 will play \( U \rightarrow \) player 2 will play \( L \)
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- Is outcome dependent on elimination order?
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- What is rational to do?
  - No matter what player 1 does: R gives player 2 a strictly higher payoff than M. „M is strictly dominated by R”
  - → player 1 knows that player 2 will not play M → U is better than M or D
  - → player 2 knows that player 1 knows that player 2 will not play M → player 2 knows that player 1 will play U → player 2 will play L
  - This elimination process: „iterated strict dominance”
  - Is outcome dependent on elimination order?

No! If s₁ is strictly worse than s₁’ against opponent’s strategy in set D then s₁ is strictly worse than s₁’ against opponent’s strategy in any subset of D

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