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### Notation: Strategic Form Games

- Set $\delta$ of players: $\{1,2,...,I\}$
  - Example: $\{1,2\}$
- Player index: $i \in \delta$
- Pure strategy space $S_i$ of player $i$
  - Example: $S_i = \{U,M,D\}$ and $S_j = \{L,M,R\}$
- Strategy profile $s = (s_1,...,s_I)$ where each $s_i \in S_i$
  - Example: $(D,M)$
- (Finite) space $S = \times_i S_i$ of strategy profiles $s \in S$
  - Example: $S = \{U,LM,DR\}$
- Payoff function $u_i: S \rightarrow \mathbb{R}$ gives von Neumann-Morgenstern-utility $u_i(s)$ for player $i$ of strategy profile $s \in S$
  - Examples: $u_i((U,L)) = 4$, $u_i((U,M)) = 3$, $u_i((M,M)) = 8$ ....
- Set of player $i$'s opponents: "-i"
  - Example: $-1 = \{2\}$
Games in Strategic Form & Nash Equilibrium

More Notation:

- Discussing player $i$’s strategy-options, holding other player’s options fixed:
  - $s_i \in S_i$; “other player’s strategies”
- Short notation: $(s'_1, s'_2) := (s_1, ..., s_i, s', s_{i+1}, ..., s_n)$
- Same for mixed strategies: $(\sigma'_1, \sigma'_2) := (\sigma_1, ..., \sigma_i, \sigma', \sigma_{i+1}, ..., \sigma_n)$

Definition:

- Pure strategy $s_i$ is strictly dominated for player $i$ if $\sigma'_i$ exists so that $u_i(\sigma'_i, s_{-i}) > u_i(s_i, s_{-i})$ for all $s_i \in S_i$
- ... weakly dominated:
  - $u_i(\sigma'_i, s_{-i}) \geq u_i(s_i, s_{-i})$ for all $s_i \in S_i$ (and > for at least one $s_i$)
- If $u_i(\sigma'_i, s_{-i}) > u_i(s_i, s_{-i})$ for all $s_i \in S_i$ we also have
  - $u_i(\sigma'_i, s_{-i}) > u_i(s_i, \sigma'_{-i})$ for all $\sigma'_{-i} \in S_{-i}$ because
  - $u_i(\sigma'_i, s_{-i})$ is a convex function of $u_i(\sigma'_i, s_{-i}), u_i(\sigma'_i, s'_{-i}), u_i(\sigma'_i, s''_{-i})$,...

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Examples:

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<thead>
<tr>
<th>L</th>
<th>R</th>
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<tbody>
<tr>
<td>U</td>
<td>1, 3</td>
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<tr>
<td>M</td>
<td>-2, 0</td>
</tr>
<tr>
<td>D</td>
<td>0, 1</td>
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Games in Strategic Form & Nash Equilibrium

- What about dominated mixed strategies?
- Easy: A mixed strategy that assigns positive probabilities to pure strategies that are dominated is dominated
- But: A mixed strategy may be dominated even if it assigns positive probabilities to pure strategies that are not even weakly dominated:

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Example:
- U and M are not dominated by D for player 1
- But: Playing $\sigma_1 = (\frac{1}{3}, \frac{1}{3}, 0)$ gives expected utility $u_1(\sigma_1^*, \sigma^*) = -1/2$ no matter what 2 plays $\Rightarrow D (\sigma_D = (0, 0, 1))$ dominates $\sigma_1$

Games in Strategic Form & Nash Equilibrium

A note on rationality

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- Iterated strict dominance $\Rightarrow (U, L)$
- BUT: psychology $\Rightarrow$ play D instead of U because "U is unsafe"
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Game Theory $\leftrightarrow$ Decision Theory

- Example

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<td>4, 1</td>
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<td>3, 4</td>
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- If player 1 reduces his payoff for U by 2:
  - decision theory: no use
  - game theory: new iterated strict dominance $\rightarrow$ (D,R)

Games in Strategic Form & Nash Equilibrium

Vickrey Auction & Iterated dominance

- Good's valuations: $v_i$; Assume common knowledge for the moment
- Bids: $s_i$
- Second price:
  - winning condition: $s_i > \max_{j \neq i} s_j$
  - let $r_i := \max_{j \neq i} s_j$; $r_i$ is the price having to be paid
  - winner i's utility: $u_i = v_i - r_i$; other players utility = 0
- for each player bidding true valuation is weakly dominant:
  - case $s_i > v_i$: (overbidding)
    - If $r_i > s_i$: looses $\rightarrow u_i = 0$
    - could have bidden $v_i$ as well
  - If $r_i \leq v_i$: wins $\rightarrow u_i = v_i - r_i$
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**Vickrey Auction & Iterated dominance**

- case $v_i < r_i < s_i$ :
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  - should have bid $v_i = r_i \Rightarrow u_i = 0$ at least
- case $s_i < v_i$ : (underbidding)
  - If $r_i \leq s_i$ or $r_i \geq v_i$ :
    - $u_i$ is unchanged if he bids $v_i$ instead of $s_i$
  - If $s_i < r_i < v_i$ :
    - bidder forgoes positive winning chances by underbidding

Assumption of common knowledge may be dropped because bidding own valuation is weakly dominant for each player

Games in Strategic Form & Nash Equilibrium

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Games in Strategic Form & Nash Equilibrium

Nash Equilibrium

- Nash Equilibrium: strategy profile: each player's strategy is optimal response to all other player's strategies:

  - Mixed strategy profile \(\sigma^*\) is Nash Equilibrium if
    for all \(i:\) \(u_i(\sigma^*_i, \sigma^{* -i}) \geq u_i(s_i, \sigma^{* -i})\) for all \(s_i \in S_i\)
    (Pure strategy profiles also possible \(\Rightarrow\) "pure strategy NE")

  - Strategy profile \(s^*\) is Strict Nash Equilibrium: if it is a NE and
    for all \(i:\) \(u_i(s^*_i, s^{* -i}) > u_i(s_i, s^{* -i})\) for all \(s_i \neq s^*_i\).
    Strict NE is necessarily a pure strategy NE by definition.
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From previous slide: $\sigma^*$ is Nash Equilibrium if for all $i$: $u_i(\sigma^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^*)$ for all $s_i \in S_i$.

Expected utilities are "linear in the probabilities"

$\rightarrow$ in NE def we must only check for pure alternatives $s_i$.

In a (non-degenerate) mixed strategy Nash Equilibrium a player must be (a priori) indifferent between all pure strategies to which he assigns positive probability (Indifference condition).

(we will analyze this in more depth later)
Nash Equilibrium

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  → In NE we must only check for pure alternatives \( s_i \)
  
  → In a (non-degenerate) mixed strategy Nash Equilibrium a player must be (a priori) indifferent between all pure strategies to which he assigns positive probability (Indifference condition)
  
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Nash Equilibrium: Example: Cournot Competition

- Cournot model: Duopoly. Each of two firms (players) i produces same good.
- Output levels \( q_i \) are chosen from sets \( Q_i \)
- Cost of production is \( c_i(q_i) \)
- Market price is \( p(q) = p(q_1 + q_2) \)
- Firm i's profit is then \( u_i(q_i, q_{-i}) = q_i p(q) - c_i(q_i) \)
- Cournot reaction functions \( r_1 : Q_2 \rightarrow Q_1 \) and \( r_2 : Q_1 \rightarrow Q_2 \) specify optimal reaction on output level of opponent

Games in Strategic Form & Nash Equilibrium

- Strict equilibria need not exist. However each finite strategy form game has a mixed strategy equilibrium.
- In NE no player has incentive to deviate from NE
- In reality: If rationality is "non-strict" (mistakes are made): deviations can occur
- If one round of elimination of strictly dominated strategies yields unique strategy profile, this strategy profile is a strict NE (unique)
- In NE positive probabilities may only be assigned to not-strictly dominated strategies (Otherwise profit may be increased by choosing a dominating strategy).
Games in Strategic Form & Nash Equilibrium

Nash Equilibrium: Example: Cournot Competition

- **Cournot model: Duopoly.** Each of two firms (players) i produces same good.
- Output levels \( q_i \) are chosen from sets \( Q_i \).
- Cost of production is \( c_i(q_i) \).
- Market price is \( p(q) = p(q_1 + q_2) \).
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Games in Strategic Form & Nash Equilibrium

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Games in Strategic Form & Nash Equilibrium

Nash Equilibrium: Example: Cournot Competition

- Under certain reasonable assumptions (see [1]) we can maximize e.g. \( u_2(q_1, q_2) \) by solving \( \frac{d}{dq_2} u_2(q_1, q_2) = 0 \) which yields \( \frac{d}{dq_2} [q_2 p(q_1, q_2) - c_2(q_2)] = p(q_1, q_2) + p'(q_1, q_2) q_2 - c_2'(q_2) = 0 \).
- Inserting \( r_2(q_1) \) for \( q_2 \):
  \[ p(q_1 + r_2(q_1)) + p'(q_1 + r_2(q_1)) r_2(q_1) - c_2'(r_2(q_1)) = 0 \]
  gives the defining equation for \( r_2(\cdot) \).
  (Analogous for \( r_1(\cdot) \).)
- The intersections of the functions \( r_2 \) and \( r_1 \) are the NE of the Cournot game.
- Example: Linear demand \( p(q) = \max(0, 1 - q) \); linear cost: \( c_i(q_i) = c q_i \);
  \[ r_2(q_1) = 1/2 (1 - q_1 - c); \quad r_1(q_2) = 1/2 (1 - q_2 - c); \]
  \[ \text{NE: } q^*_2 = r_2(q^*_1) = 1/3 (1 - c) = q^*_1 = r_1(q^*_2) \]
Nash Equilibrium: Example: Cournot Competition

Under certain reasonable assumptions (see [1]) we can maximize e.g. \( u_2(q_1, q_2) \) by solving \( \frac{d}{dq_2} u_2(q_1, q_2) = 0 \) which yields

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\frac{d}{dq_2} [q_2 p(q_1, q_2) - c_2(q_2)] = p(q_1, q_2) + p'(q_1, q_2) q_2 - c_2'(q_2) = 0.
\]

Inserting \( r_2(q_1) \) for \( q_2 \)

\[
p(q_1 + r_2(q_1)) + p'(q_1 + r_2(q_1)) r_2(q_1) - c_2'(r_2(q_1)) = 0
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Example: Linear demand \( p(q) = \max(0, 1-q) \); linear cost: \( c_i(q_i) = c q_i \):

\[
\rightarrow r_2(q_1) = \frac{1}{2} (1 - q_1 - c); \quad r_1(q_1) = \frac{1}{2} (1 - q_2 - c);
\]

\[
\rightarrow \text{NE: } q^*_2 = r_2(q^*_1) = 1/3 (1-c) = q^*_1 = r_1(q^*_2)
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Games in Strategic Form & Nash Equilibrium

Nash Equilibrium: Example: Cournot Competition

- Under certain reasonable assumptions (see [1]) we can maximize e.g. $u_2(q_1, q_2)$ by solving $d/dq_2 \ u_2(q_1, q_2) = 0$ which yields
  
  $d/dq_2 \ [q_2 \ p(q_1, q_2) - c(q_2)] = p(q_1, q_2) + p'(q_2, q_2) q_2 - c'_2(q_2) = 0.$

  Inserting $r_2(q_1)$ for $q_2$
  
  $p(q_1 + r_2(q_1)) + p'(q_2 + r_2(q_2)) r_2(q_2) - c'_2(r_2(q_2)) = 0$

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- Example: Linear demand $p(q) = \max(0, 1-q)$; linear cost: $c_i(q_i) = c q_i$:
  
  $r_2(q_2) = 1/2 (1- q_1 - c); \ r_1(q_1) = 1/2 (1- q_2 - c);$

  $\Rightarrow NE: q_2^* = 1/3 (1-c); q_1^* = 1/3 (1-c).$

Games in Strategic Form & Nash Equilibrium

Nash Equilibrium: Non-Existence-of Pure NE-Example

- Some games may have more than one pure strategy NE

- Not all games have a pure strategy NE:

  - Example: Matching pennies:

    |   | H | T |
    |---|---|---|
    | H | 1, -1 | -1, 1 |
    | T | -1, 1 | 1, -1 |

- Reasoning: If player 2 plays $(1/2, 1/2)$ then player 1’s expected payoff is $1/2 \cdot 1 + 1/2 \cdot (-1) = 0$ when playing $H$ and $1/2 \cdot 1 + 1/2 \cdot 1 = 0$ when playing $T$ so player 1 is also indifferent.

Games in Strategic Form & Nash Equilibrium

Nash Equilibrium: Non-Existence--of Pure NE-Example 2

- Another example: Inspection game

  - Worker: work or shirk; Employer: Inspect or not inspect

    |   | I | NI |
    |---|---|---|
    | S | 0, -h | w, -w |
    | W | w, v-h | w, v-w |

- Reasoning: If worker 2 does not inspect then worker 1’s expected payoff is $w \cdot 1 + v \cdot 1 = w$ when playing $S$ and $v \cdot 1 = v$ when playing $W$.
Nash Equilibrium: Non-Existence--of Pure NE-Example 2

- Another example: Inspection game
  - Worker: work or shirk; Employer: Inspect or not inspect
    - Worker: working costs $g$, produces value $v$; gets wage $w$
      - Employer: Inspection costs $h$
      - We assume $w > g > h > 0$
      - If not inspect → worker shirks → better inspect → if inspect → worker always works → better not inspect → ...: No pure NE
      - Employer must randomize

---

If worker plays $(x, 1-x)$ and employer plays $(y, 1-y)$

Indifference condition in mixed strategy NE →

- For worker indifferent between $S$ and $W$:
  - gain from shirking == expected income loss:
    \[
    0y + (1-y)w = y(w-g) + (1-y)(w-g)
    \]
  - $g = yw 
    \[
    g = yw \rightarrow y = g/w
    \]

- For employer indifferent between $I$ and $NI$:
  - inspection costs == expected wage savings:
    \[
    x(-h)+ (1-x)(v-w-h) = x(-w) + (1-x)(v-w)
    \]
  - $h = xw \rightarrow x = h/w

---

Games in Strategic Form & Nash Equilibrium
### Nash Equilibrium: Non-Existence--of Pure NE-Example 2

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- **Indifference condition** in mixed strategy NE →
  - For worker indifferent between \(S\) and \(W\):
gain from shirking = expected income loss:
  \[0y + (1-y)w = y(w-g) + (1-y)(w-g)\]
  \[\Rightarrow g = yw \Rightarrow y = g/w\]
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### Games in Strategic Form & Nash Equilibrium

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Games in Strategic Form & Nash Equilibrium

Nash Equilibrium: Non-Existence—of Pure NE—Example 2

- If worker plays \( x, 1-x \) and employer plays \( y, 1-y \)
- Indifference condition in mixed strategy NE →
  - \( g = yw \rightarrow y = g/w \)
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Games in Strategic Form & Nash Equilibrium

Nash Equilibrium: More than one NE

- Another example: Battle of the sexes
- Man & Woman; Ballet or Football

Games in Strategic Form & Nash Equilibrium

Nash Equilibrium: Non-Existence—of Pure NE—Example 2

- If worker plays \( x, 1-x \) and employer plays \( y, 1-y \)
- Indifference condition in mixed strategy NE →
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### Nash Equilibrium: More than one NE

#### Another example: Battle of the sexes

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<td>B</td>
<td>1, 2</td>
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#### Another example: Game of chicken

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<tr>
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</tr>
<tr>
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<td>1, 2</td>
<td>0, 0</td>
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### Nash Equilibrium: More than one NE

#### Another example: Battle of the sexes

- Two pure NE: (F,F) and (B,B)
- One mixed NE: Indifference condition
  - Let \( \sigma_1(F) = x \) and \( \sigma_1(B) = y \)
  - Player 1’s indifference:
    - \( 0 + 2(1-y) = 1 y + 0(1-y) \) → \( y = \frac{2}{3} \)
  - Player 2’s indifference:
    - \( 0 x + 2(1-x) = 1 x + 0(1-x) \) → \( x = \frac{2}{3} \)
  - Mixed NE: ((2/3, 1/3); (2/3, 1/3))

#### Another example: Game of chicken

- (same reasoning) → Mixed NE: ((1/2, 1/2); (1/2, 1/2))
**Games in Strategic Form & Nash Equilibrium**

**Nash Equilibrium: More than one NE**

- **Another example: Battle of the sexes**
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- One mixed NE: Indifference condition
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  \[ 0 \cdot y + 2(1-y) = 1 \cdot y + 0(1-y) \Rightarrow y = 2/3 \]
  Player 2's indifference:
  \[ 0 \cdot x + 2(1-x) = 1 \cdot x + 0(1-x) \Rightarrow x = 2/3 \]
  Mixed NE: \( (2/3, 1/3); (2/3, 1/3) \)

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**Games in Strategic Form & Nash Equilibrium**

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- **Another example: Game of chicken**
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**Focal points**

- Some games have more than one NE → which will be chosen?
- Theory of „focalness“ of NE („focal points“): Example: Chose time of day simultaneously; reward if match: 12 noon is focal, 15:37 is not

**Risk Dominance**

- Stag Hunt: NE: \( (C,C) \) and \( (D,D) \); \( (C,C) \) is pareto-dominant → \( (C,C) \) might be chosen if \( p(C)>0.5 \) BUT
- more than two players: ALL have to agree on \( C \) → \( p(C) > 0.5 \) → \( p(C) > 0.93 \) → \( (D,D) \) „risk dominates“ \( (C,C) \)
Games in Strategic Form & Nash Equilibrium

Nash Equilibrium: More than one NE

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Risk Dominance

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<table>
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<tr>
<th>Hunt</th>
<th>Stag (C)</th>
<th>Hare (D)</th>
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<tr>
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Games in Strategic Form & Nash Equilibrium

Nash Equilibrium: More than one NE

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### Games in Strategic Form & Nash Equilibrium

#### Nash Equilibrium: More than one NE

**Risk Dominance / Pareto Optimality**

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<tbody>
<tr>
<td>U</td>
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<td>-5,-5,0</td>
</tr>
<tr>
<td>D</td>
<td>-5,-5,0</td>
<td>1,1,-5</td>
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</table>

- Three player game: Two pure NE: (U,L,A) and (D,R,B); (and one mixed) ; (U,L,A) pareto-dominates (D,R,B)
- If player 3’s choice is fixed → Two player game → (D,R) is pareto-dominant → if players 1 and 2 expect A: coordinate on (D,R).
- → concept of “coalition proof eq.” (here (D,R,B))(see [1])

#### Mixed Nash Equilibrium: General Analysis for 2 x 2 Games

(see [2])

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<tr>
<td>1-q</td>
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</tbody>
</table>

- **Pure NE:** One cell → For A: cell’s payoff for A must be (weak) maximum over rows in that column For B: cell’s payoff for B must be (weak) maximum over column in that row
- **Example:** (U,R) is pure NE if \( a_{UR} \geq a_{DL} \) and \( b_{UR} \geq b_{UL} \)