- Basic scenario: Players simultaneously choose action to perform → result of the actions they select → outcome in discrete state space $\Omega$
- outcome depends on the combination of actions
- Assume: each player has just two possible actions $C$ ("cooperate") and $D$ ("defect")
- Environment behavior given by state transformer function:
  $$\tau : \mathcal{A}_C \times \mathcal{A}_C \rightarrow \Omega$$
  Player $i$'s action  Player $j$'s action

Examples for state transformer function

- $\tau(D, D) = \omega_1  \quad \tau(D, C) = \omega_2  \quad \tau(C, D) = \omega_3  \quad \tau(C, C) = \omega_4$
  (environment is sensitive to actions of both players)
- $\tau(D, D) = \omega_1  \quad \tau(D, C) = \omega_1  \quad \tau(C, D) = \omega_1  \quad \tau(C, C) = \omega_1$
  (Neither player has any influence in this environment.)
- $\tau(D, D) = \omega_1  \quad \tau(D, C) = \omega_2  \quad \tau(C, D) = \omega_1  \quad \tau(C, C) = \omega_2$
  (environment is controlled by $j$.)
Examples for state transformer function

- \( \tau(D, D) = \omega_1 \quad \tau(D, C) = \omega_2 \quad \tau(C, D) = \omega_3 \quad \tau(C, C) = \omega_4 \)
  (environment is sensitive to actions of both players)

- \( \tau(D, D) = \omega_1 \quad \tau(D, C) = \omega_1 \quad \tau(C, D) = \omega_1 \quad \tau(C, C) = \omega_1 \)
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- \( \tau(D, D) = \omega_1 \quad \tau(D, C) = \omega_2 \quad \tau(C, D) = \omega_1 \quad \tau(C, C) = \omega_2 \)
  (environment is controlled by \( j \).)

Rational Behavior

- **Assumption:** Environment is sensitive to actions of both players:
  \( \tau(D, D) = \omega_1 \quad \tau(D, C) = \omega_2 \quad \tau(C, D) = \omega_3 \quad \tau(C, C) = \omega_4 \)

- Assumption:
  \( u_i(\omega_1) = 1 \quad u_i(\omega_2) = 1 \quad u_i(\omega_3) = 4 \quad u_i(\omega_4) = 4 \)

  Utility functions:
  \( u_i(\omega_1) = 1 \quad u_i(\omega_2) = 4 \quad u_i(\omega_3) = 1 \quad u_i(\omega_4) = 4 \)

- **Short notation:**
  \( u_i(D, D) = 1 \quad u_i(D, C) = 1 \quad u_i(C, D) = 4 \quad u_i(C, C) = 4 \)

  \( u_j(D, D) = 1 \quad u_j(D, C) = 4 \quad u_j(C, D) = 1 \quad u_j(C, C) = 4 \)

- **\( \rightarrow \) player’s preferences:**
  (also in short notation):
  \( C, C \gtrsim_i C, D \quad \gtrsim_i D, C \gtrsim_i D, D \)
Rational Behavior

\[
\begin{align*}
    u_i(D, D) &= 1 & u_i(D, C) &= 1 & u_i(C, D) &= 4 & u_i(C, C) &= 4 \\
    u_j(D, D) &= 1 & u_j(D, C) &= 4 & u_j(C, D) &= 1 & u_j(C, C) &= 4
\end{align*}
\]

\[
C, C \succeq_i C, D \quad \succ_i D, C \succeq_i D, D \\
C, C \succeq_j D, C \quad \succ_j C, D \succeq_j D, D
\]

- "C" is the rational choice for i.
  (Because i (strongly) prefers all outcomes that arise through C over all outcomes that arise through D.)

- "C" is the rational choice for j.
  (Because j (strongly) prefers all outcomes that arise through C over all outcomes that arise through D.)

Dominant Strategies and Nash Equilibria

- With respect to "what should I do":
  If \( \Omega = \Omega_1 \cup \Omega_2 \) we say \( \Omega_1 \) weakly dominates \( \Omega_2 \) for player i if for player i every state (outcome) in \( \Omega_1 \) is preferable to or at least as good as every state in \( \Omega_2 \):

\[
\forall \omega_1 \forall \omega_2 : (\omega_1 \in \Omega_1 \land \omega_2 \in \Omega_2) \rightarrow \omega_1 \succeq_i \omega_2
\]

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\[
\forall \omega_1 \forall \omega_2 : (\omega_1 \in \Omega_1 \land \omega_2 \in \Omega_2) \rightarrow \omega_1 \succ_i \omega_2
\]

Example:
\[
\begin{align*}
    \Omega &= \{\omega_1, \omega_2, \omega_3, \omega_4\} & \Omega_1 &= \{\omega_1, \omega_2\} \\
    \omega_1 \succ_i \omega_2 \succ_i \omega_3 \succ_i \omega_4 & \Omega_2 &= \{\omega_3, \omega_4\}
\end{align*}
\]

\( \Omega_1 \) strongly dominates \( \Omega_2 \) for player i.

---

Game theory notation: actions are called "strategies"

- Notation: \( s \) is the set of possible outcomes (states) when "playing strategy \( s \)" (executing action \( s \))

Example: if we have (as before):

\[
\tau(D, D) = \omega_1 \quad \tau(D, C) = \omega_2 \quad \tau(C, D) = \omega_3 \quad \tau(C, C) = \omega_4
\]

we have (from player i's point of view):

\[
D^* = \{\omega_1, \omega_2\} \quad C^* = \{\omega_3, \omega_4\}
\]

- Notation: "strategy \( s1 \) (strongly / weakly) dominates \( s2 \)" iff \( s1^* \) (strongly / weakly) dominates \( s2^* \)

- If one strategy strongly dominates the other → question what to do is easy. (do first)
The Prisoner’s Dilemma

- Two criminals are held in separate cells (no communication):
  1. One confesses and the other does not → confessor is freed and the other gets 3 years
  2. Both confess → each gets 2 years
  3. Neither confesses → both get 1 year

- Associations: Confess == D; Not Confess == C

- Payoff matrix

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<tr>
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Take place of prisoner (e.g., prisoner i) → Course of Reasoning:

- Suppose I cooperate: If j also cooperates → we both get payoff 3. If j defects → I get payoff 0. Best guaranteed payoff when I cooperate is 0

- Suppose I defect: If j cooperates → I get payoff 5. If j also defects → both get payoff 2. Best guaranteed payoff when I defect is 2

- If I defect I'll get a minimum guaranteed payoff of 2. If I cooperate I'll get a minimum guaranteed payoff of 0.

- If prefer guaranteed payoff of 2 to guaranteed payoff of 0.

- I should defect
The Prisoner’s Dilemma

\[ u_i(D,D) = 2, \quad u_i(D,C) = 5, \quad u_i(C,D) = 0, \quad u_i(C,C) = 3 \]
\[ u_j(D,D) = 2, \quad u_j(D,C) = 5, \quad u_j(C,D) = 0, \quad u_j(C,C) = 3 \]

\[(D,C) \succ_i (C,C) \succ_j (D,D) \succ_j (C,D)\]
\[(C,D) \succ_i (C,C) \succ_j (D,D) \succ_j (D,C)\]

- **Take place of prisoner (e.g. prisoner i)**
  - Course of Reasoning:
    - Suppose I cooperate: If j also cooperates → we both get payoff 3. If j defects → I get payoff 0. Best guaranteed payoff when I cooperate is 2.
    - Suppose I defect: If j cooperates → I get payoff 5. If j also defects → both get payoff 2. Best guaranteed payoff when I defect is 2.
    - If I defect I’ll get a minimum guaranteed payoff of 2. If I cooperate I’ll get a minimum guaranteed payoff of 0.
    - If prefer guaranteed payoff of 2 to guaranteed payoff of 0. I should defect.

The Prisoner’s Dilemma

\[ u_i(D,D) = 2, \quad u_i(D,C) = 5, \quad u_i(C,D) = 0, \quad u_i(C,C) = 3 \]
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\[(D,C) \succ_i (C,C) \succ_j (D,D) \succ_j (C,D)\]
\[(C,D) \succ_i (C,C) \succ_j (D,D) \succ_j (D,C)\]

- Only one Nash equilibrium: \((D,D)\). (Under the assumption that the other does D, one can do no better than do D^*)
- Intuition says: \((C,C)\) is better than \((D,D)\) so why not \((C,C)\)?
  - But if player assumes that other player does C it is BEST to do D! → seemingly „waste of utility“

- Shocking „truth“: defect is rational, cooperate is irrational.
- Other prisoner’s dilemma: Nuclear arms reduction (D: do not reduce, C: reduce)

The Prisoner’s Dilemma

- Two criminals are held in separate cells (no communication):
  1. One confesses and the other does not → confessor is freed and the other gets 3 years.
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„sucker’s payoff“
Other symmetric 2x2 Games

Stag Hunt

• Going back to J.J. Rousseau (1775)

• Modern variant: You and a friend decide: good joke to appear both naked on a party. C: really do it; D: not do it

\[(C, C) \succ_i (D, C) \succ_i (D, D) \succ_i (C, D)\] 

\[
\begin{array}{c|cc}
   & i:D & i:C \\
\hline
j:D & 1 & 2 \\
j:C & 2 & 3 \\
\end{array}
\]

• Two Nash equilibria: (D,D), (C,C)
   (Assuming the other does D you can do no better than do D
   Assuming the other does C you can do no better than do C)

Other symmetric 2x2 Games

Game of Chicken

• Going back to a James Dean film

• Modern variant: Gangster and hero drive cars directly towards each other C: steer away; D: not steer away

\[(D, C) \succ_i (C, C) \succ_i (C, D) \succ_i (D, D)\] 

\[
\begin{array}{c|cc}
   & i:D & i:C \\
\hline
j:D & 0 & 3 \\
j:C & 1 & 2 \\
\end{array}
\]

• Two Nash equilibria: (D,C), (C,D)
   (Assuming the other does D you can do no better than do C
   Assuming the other does C you can do no better than do D)

Notation: Strategic Form Games

- Set \(\delta\) of players: \(\{1, 2, \ldots, l\}\)
  Example: \(\{1, 2\}\)

- Player index: \(i \in \delta\)

- Pure strategy space \(S_i\) of player \(i\)
  Example: \(S_1 = \{U,M,D\}\) and \(S_2 = \{L,M,R\}\)

- Strategy profile \(s = (s_1, \ldots, s_l)\) where each \(s_i \in S_i\)
  Example: \((D,M)\)

- (Finite) space \(S = \times_i S_i\) of strategy profiles \(s \in S\)
  Example: \(S = \{(U,L), (U,M), \ldots, (D,R)\}\)

- Payoff function \(u_i: S \rightarrow \mathbb{R}\) gives von Neumann-Morgenstern-utility \(u_i(s)\) for player \(i\) of strategy profile \(s \in S\)
  Examples: \(u_1((U,L)) = 4\), \(u_1((L,L)) = 3\), \(u_1((M,M)) = 8\) ....

- Set of player \(i\)'s opponents: \(\sim i\)
  Example: \(-i = \{2\}\)
Notation: Strategic Form Games

- Set $\mathcal{S}$ of players: \{1, 2, ..., $n$\}
  - Example: \{1, 2\}

- Player index: $i \in \mathcal{S}$

- Pure Strategy Space $S_i$ of player $i$
  - Example: $S_i = \{L, M, D\}$ and $S_j = \{L, M, R\}$

- Strategy profile $s = (s_1, ..., s_n)$ where $s_i \in S_i$
  - Example: $(D, M)$

- (Finite) space $S = X \times S_i$ of strategy profiles $s \in S$
  - Example: $S = \{ (U, L), (U, M), ..., (D, R) \}$

- Payoff function $u_i: S \to \mathbb{R}$ gives von Neumann-Morgenstern-utility $u_i(s)$ for player $i$ of strategy profile $s \in S$
  - Examples: $u_i((U, L)) = 4$, $u_i((U, L)) = 3$, $u_i((M, M)) = 8$, ...

- Set of player $i$'s opponents: "-i"
  - Example: $-1 = \{2\}$

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- Set of player $i$'s opponents: "-i"
  - Example: $-1 = \{2\}$
**Notation: Strategic Form Games**

- **Two Player zero sum game:**
  \[ \forall s : \sum_{i=1}^{2} u_i(s) = 0 \]

- **Structure of game is common knowledge:**
al all players know;
al all players know that all players know;
al all players know that all players know that all players know;

- **Mixed strategy** \( \sigma_i : S_i \rightarrow [0,1] \) Probability distribution over pure strategies (statistically independent for each player);
  Examples: \( \sigma_i(U) = 1/3, \sigma_i(M) = 2/3, \sigma_i(D) = 0 \);
  \( \sigma_i'(U) = 2/3, \sigma_i'(M) = 1/6, \sigma_i'(D) = 1/6 \);

<table>
<thead>
<tr>
<th></th>
<th>U</th>
<th>M</th>
<th>R</th>
</tr>
</thead>
</table>
  L | 4.3 | 5.1 | 6.2 |
  M | 2.1 | 8.4 | 3.6 |
  D | 3.0 | 9.6 | 2.8 |

Thus: \( \sigma_i(s_i) \) is the probability that player \( i \) assigns to strategy (action) \( s_i \).

---

**Sense of Mixed Strategy Concept**

- **Example: Rock Paper Scissors**

<table>
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<tr>
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<th>Paper</th>
<th>Scissors</th>
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<tr>
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<td>-1.1</td>
<td>1, 1</td>
</tr>
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<td>-1.1</td>
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<td>-1.1</td>
</tr>
<tr>
<td>Scissors</td>
<td>-1.1</td>
<td>1, -3</td>
<td>0, 0</td>
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- no pure NE, but mixed NE if both play (1/3, 1/3, 1/3)
• Space of mixed strategies for player $i$: $\sum_i$

• Space of mixed strategy profiles: $\Sigma = x_i \Sigma_i$

• Mixed strategy profile $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_i) \in \Sigma$

• Player $i$'s payoff when a mixed strategy profile $\sigma$ is played is

$$\sum_{\sigma \in \Sigma} \left( \prod_{j=1}^{f} \sigma_j(s_j) \right) u_i(s)$$

denoted as $u_i(\sigma)$, is a linear function of the $\sigma_i$

• A pure strategy of a player is a special mixed strategy of that player with one probability equal to 1 and all others equal to 0
Notation: Strategic Form Games

Example:
Let
\[
\sigma_1(U) = 1/3, \quad \sigma_1(M) = 1/3, \quad \sigma_1(D) = 1/3
\]
\[
\sigma_2(L) = 0, \quad \sigma_2(M) = 1/2, \quad \sigma_2(R) = 1/2
\]
or short
\[
\sigma_1 = (1/3, 1/3, 1/3)
\]
\[
\sigma_2 = (0, 1/2, 1/2)
\]
We then have:
\[
u_1(\sigma_1, \sigma_2) = \frac{1}{3} (0 \cdot 4 + \frac{1}{2} \cdot 5 + \frac{1}{2} \cdot 6)
+ \frac{1}{3} (0 \cdot 2 + \frac{1}{2} \cdot 8 + \frac{1}{2} \cdot 3)
+ \frac{1}{3} (0 \cdot 3 + \frac{1}{2} \cdot 9 + \frac{1}{2} \cdot 2) = 11/2
\]
\[
u_2(\sigma_1, \sigma_2) = \text{...} = 27/6
\]

Games in Strategic Form & Nash Equilibrium

- What is rational to do?
  - No matter what player 1 does: R gives player 2 a strictly higher payoff than M.
    „M is strictly dominated by R“
  - \(\rightarrow\) player 1 knows that player 2 will not play M \(\rightarrow\) U is better than M or D
  - \(\rightarrow\) player 2 knows that player 1 knows that player 2 will not play M \(\rightarrow\) player 2 knows that player 1 will play U \(\rightarrow\) player 2 will play L
  - This elimination process: „iterated strict dominance“
Games in Strategic Form & Nash Equilibrium

- **New example:**
  - Player 1: M not dominated by U and M not dominated by D
  - But: If Player 1 plays \( \sigma_1 = (1/2, 0, 1/2) \) he will get \( u(\sigma_1) = 1/2 \) regardless how player 2 plays
  - → a pure strategy may be dominated by a mixed strategy even if it is not strictly dominated by any pure strategy

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