Distances: Eccentricity

- **Example:** Facility location problems: Objective function on $d(u,v)$: e.g. minimax (minimize maximal distance (e.g. hospital emergency)) → can be mapped to social case

- For the moment: $G$ is **undirected and unweighted** (e.g. "friendship"). Mapping to weighted and / or directed case is possible.

- Eccentricity $e(u) = \max\{d(u,v); v \in V\}$

Distances: Centroids

- **Competitive objective:** Given number of competitors: where to open a store (Customers will just choose store based on minimal distance)?

- **Social Problem:** Example: find "social ecological niche" (do computer scientists try to find a partner at a computer science party or at social science parties?) 😄

- **Formalization:** For $u, v : \gamma^t(v)=$number of vertices closer to $u$ than to $v$; if one salesman selects $u$ and competitor selects $v$ as locations, the first will have

  $\gamma^t(v) + \frac{1}{2} ( |V| - \gamma^t(v) - \gamma^t(u)) = \frac{1}{2} |V| + \frac{1}{2} (\gamma^t(v) - \gamma^t(u))$

  customers

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customers

\[ f(u,v) = \gamma_u(v) - \gamma_v(u) \]

**Possible centrality index:** First salesman knows the strategy of the competitor and calculates for each location the worst case:

\[ c(u) = \min_{v} \{ f(u,v) : v \in V \setminus \{u\} \} \]

\( c(u) \) is called centroid value: measures the advantage of location \( u \) compared to other locations: Minimal loss of customers if he choses \( u \) and a competitor choses \( v \)
**Distances: Centroids**

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**Shortest Paths: Stress**

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- **Heuristic**: If a vertex is part of many shortest paths \( \rightarrow \) "much information will run through it" if information is routed along shortest paths

- **Social analogon**: People that are asked to contribute to a workflow more often than others

- \( \rightarrow \) A vertex \( v \) is more central the more shortest paths run through it. Let \( \sigma_{ab}(v) \) denote the number of shortest paths from node a to node b containing \( v \). \( \sigma_{ab}(v) \) can be \( >1 \) if there there are several paths with the same minimal length

  - **stress centrality**:
    \[
    c(v) = \sum_{a \notin v, b \in v} \sum_{s \in V} \sigma_{ab}(v)
    \]
**Shortest Paths: Stress**

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  \[
  c(v) = \sum_{a \in V, a \neq v} \sum_{b \in V, b \neq v} \frac{\sigma_{ab}(v)}{\sigma_{ab}}
  \]

**Interpretation:** Probability that \( v \) is involved in a communication between \( a \) and \( b \)

**Shortest Paths: Shortest Path Betweenness**

- **Shortest Path Betweenness (SPB) centrality** is then:

  \[
  c(v) = \sum_{a \in V} \sum_{b \in V} \delta_{ab}(v)
  \]

- **Interpretation:** Control that \( v \) exceeds on the communication in the graph

- Also applicable to disconnected graphs

- Algorithm by Ulrik Brandes computes SPB in \( O(|V||E| + |V|^2 \log |V|) \) time

**Possible:** Interpret quantity \( \delta_{ab}(v) \) as general relative information flow through \( v \) (“rush”)

**Other variants:** Instead of shortest paths between \( a \) and \( b \) regard
- the set of all paths
- the set of the \( k \)-shortest paths (interesting for social case; choose small \( k \))
- the set of the \( k \)-shortest node disjoint paths
- the set of paths not longer than \( (1+r)d(a,b) \)

**Shortest Paths: Shortest Path Betweenness**

- Define \( c_{SPB} \) for edges analogously

  \[
  c(e) = \sum_{a \in V} \sum_{b \in V} \delta_{ab}(e)
  \]
What we have seen so far: Various centrality measures mostly for vertices (based on degree, closeness, betweenness).

→ Formal way to translate a given vertex centrality index to a corresponding edge centrality: Apply the given vertex centrality to a transformed version of \( G \), the edge graph.

Given original \( G = (V,E) \) then the edge graph \( G' = (E,K) \) is defined by taking original edges as vertices. Two original edges are connected in \( G' \) if they are originally incident to the same original node.

Size of \( G' \) may be quadratic (w.r.t. number of nodes) compared to \( G \).

Remember: Vertex stress centrality for node \( x \): Number of shortest paths that use \( x \); Straightforward version for edge \( e \): Number of shortest paths that use \( e \);

→ Upper Example: \( G \): Stress centrality of edge \( a \) would be 3; But in edge graph \( G' \) stress centrality of original edge \( a \) (now a node) is 0.

→ Formal translations of vertex centrality indices to edge centralities with edge graphs are not well suited for all purposes.

→ Introduce incidence graph \( G'' \): Each original edge is split by new “edge vertex” that represents the edge \( \rightarrow \) Now vertex indices can be applied, preserving the intuition.
Deriving edge centralities from vertex centralities

- **Remember**: Vertex stress centrality for node \( x \): Number of shortest paths that use \( x \); Straightforward version for edge \( e \): Number of shortest paths that use \( e \);

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- Introduce incidence graph \( G'' \): Each original edge is split by new "edge vertex" that represents the edge  →  Now vertex indices can be applied, preserving the intuition.

Vitality

- **Intuition**: Measure importance of vertex (or edge) by the difference of a given quality measure \( q \) on \( G \) with or without the vertex (edge):

- **Example 1 for quality measure \( q \):** Flow:
  
  Given directed graph \( G \) with positive edge weights \( w \) modeling capacities. The flow \( f(s, t) \) from node \( s \) (source) to node \( t \) (sink) is defined as:

  \[
  f(s, t) = \sum_{e \in \text{Out-Edges of } s} \tilde{f}(e) - \sum_{e \in \text{In-Edges of } t} \tilde{f}(e)
  \]

  where the local flows \( \tilde{f} \) respect capacity constraints: \( 0 \leq \tilde{f}(e) \leq w(e) \) and balance conditions:

  \[
  \forall v \in V \setminus \{s, t\} : \sum_{e \in \text{Out-Edges of } v} \tilde{f}(e) = \sum_{e \in \text{In-Edges of } v} \tilde{f}(e)
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Vitality

• Intuition: Measure importance of vertex (or edge) by the difference of a given quality measure \( q \) on \( G \) with or without the vertex (edge):

  \[ v(x) = q(G) - q(G \setminus x) \]

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Intuition: Measure importance of vertex (or edge) by the difference of a given quality measure \( q \) on \( G \) with or without the vertex (edge):

\( \rightarrow \) Vitality \( v(x) \) of graph element \( x \) : \( v(x) = q(G) - q(G\setminus{x}) \)

Example 1 for quality measure \( q \): Flow:

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\]
• Computing a flow $f : E \rightarrow \mathbb{R}$ of maximum value (tweaking the local flows): $O(|V| |E| \log(|V|^2 |E|))$ (Algorithm by Goldberg & Tarjan (see [2]))

• Now define quality measure by e.g.:  
  
  $q(G) = \sum_{i \in V} \max_{s,t \in E} f(s, t)$

• Social analog of flow: Workflow, Information-flow, “Doing favors flow” etc.

• Besides vitality-based centrality $c(x) = v(x) = q(G) - q(G \setminus \{x\})$ we may also define a centrality as max-flow betweenness: denote: $f_{st}(G) = \max_{f(s,t)}$ we may then define:

  \[ c(u) = \sum_{s,t \in V \setminus u \setminus s,t} \frac{f_{st}(G) - f_{st}(G \setminus \{u\})}{f_{st}(G)} \]

• The numerator denotes the amount of flow that must go through node $u$

• Example 2: Mobile (Peer to Peer) communication-network: Each node should be connected to each other node by as few intermediaries as possible. \implies quality measure: Wiener Index

  \[ q(G) = \sum_{v \in V} \sum_{w \in V} d(v, w) \]

• Possible: write Wiener Index with the help of closeness centrality $c_c(v)$

  \[ q(G) = \sum_{v \in V} c_c(v) \]

• Define centrality “closeness vitality” of graph element $x$ as vitality:

  \[ c(x) = q(G) - q(G \setminus \{x\}) \]
Vitality

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  \[ q(G) = \sum_{v \in V} \sum_{w \in V} d(v, w) \]

- Possible: write Wiener Index with the help of closeness centrality \( c_c(v) \)
  \[ q(G) = \sum_{v \in V} \frac{1}{c_c(v)} \]

- Define centrality "closeness vitality" of graph element \( x \) as vitality:
  \[ c(x) = q(G) - q(G \setminus \{x\}) \]

Stress Centrality as Vitality

- We had: stress centrality of \( v \) or \( e \) is equal to number of shortest paths through \( v \) or \( e \)
  \[ c_{stress}(v) = \sum_{a \in V \setminus x} \sum_{b \in V \setminus x} \sigma_{ab}(v) \quad c_{stress}(e) = \sum_{a \in V \setminus x} \sum_{b \in V \setminus x} \sigma_{ab}(e) \]

- Intuition: \( c_{stress}(v) \) seems to measure the number of shortest paths that would be lost if \( v \) wasn't available anymore

- Why can't we directly use \( c_{stress} \) as a graph quality index to construct a vitality index ?

- Because actual number of shortest paths can INCREASE if e.g. edge is taken away

Stress Centrality as Vitality

- **Example**

  ![Diagram](image1)

  - Number of shortest paths: 54
  - Number of shortest paths containing \( e \): 8
  - \( \sigma_{ed} = 1 \) (length 3)

  ![Diagram](image2)

  - Number of shortest paths: 64
  - (18 of them have increased in length)
  - \( \sigma_{cd} = 4 \) (length 4)

- **Example**

  ![Diagram](image3)

  - Number of shortest paths: 54
  - Number of shortest paths containing \( e \): 8
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  ![Diagram](image4)

  - Number of shortest paths: 64
  - (18 of them have increased in length)
  - \( \sigma_{cd} = 4 \) (length 4)
Stress Centrality as Vitality

- In order to define a vitality-like version of stress: Consider only those shortest paths that haven't changed their length:

\[ c_{\text{vitality}}(v, G) = c_{\text{stress}}(v, G) - c_{\text{stress}}(v, G \setminus \{v\}) \]

with

\[ c_{\text{stress}}(v, G \setminus \{v\}) = \sum_{a \in V} \sum_{b \in V} \sigma_{ab}[d_G(a, b) = d_{G \setminus \{v\}}(a, b)] \]

(Iverson notation)

Critique on Betweenness Based Centralities

- major critique: Max-Flow betweenness centrality (suggested to counteract this drawback) may exhibit similar problems

- here: special Max-Flow betweenness centrality mfb:
  -- limit edge capacity to one
  -- mfb(i) := maximum possible flow through i over all possible solutions to the s-t-maximum flow problem, averaged over all s and t.

(b) In calculations of flow betweenness, vertices A and B in this configuration will get high scores while vertex C will not.

Source: [5]

Critique on Betweenness Based Centralities

Random Walk Centrality == Current Flow Btw. Centrality (see [5])

- flow of electric current in a resistor network; 
  \( V_i \) = voltage (potential) at vertex i

- \( \leftrightarrow \) Current Flow betweenness cfb centrality: cfb(i) := amount of current that flows through i in this setup, averaged over all s and t.

(b) In calculations of flow betweenness, vertices A and B in this configuration will get high scores while vertex C will not.

Source: [5]
- Flow of electric current in a resistor network; 
  $V_i = \text{voltage (potential) at vertex } i$

- Current Flow Betweenness Centrality: $c_{fb}(i) = \text{amount of current that flows through } i \text{ in this setup, averaged over all } s \text{ and } t.$

- Kirchhoff's point law (current conservation): total current flow in / out of node is zero:
  $\sum_j A_{ij}(V_i - V_j) = \delta_{is} - \delta_{it},$
  $A_{ij} = \begin{cases} 1, & \text{if there is an edge between } i \text{ and } j, \\ 0, & \text{otherwise}, \end{cases}$
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One unit of current in

One unit of current out

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**Random Walk Centrality == Current Flow Btw. Centrality (see [5])**

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\[ \sum_j A_{ij} = k_i, \text{ the degree of vertex } i \]

\[ \sum_j A_{ij}(V_i - V_j) = \delta_{is} - \delta_{it} \quad \rightarrow \quad (D - A) \cdot \mathbf{V} = \mathbf{s} \]

“Graph Laplacian”

\( D \) is the diagonal matrix with elements \( D_{ii} = k_i \)

Source vector \( \mathbf{s} \)

\[ s_i = \begin{cases} +1 & \text{for } i = s, \\ -1 & \text{for } i = t, \\ 0 & \text{otherwise}. \end{cases} \]

\[ \mathbf{V} = (D - A)^{-1} \cdot \mathbf{s} \]
Random Walk Centrality == Current Flow Btw. Centrality (see [5])

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source vector \( s \) \( s_i = \begin{cases} +1 & \text{for } i = s, \\ -1 & \text{for } i = t, \\ 0 & \text{otherwise.} \end{cases} \)

\[ V = (D_v - A_v)^{-1} \cdot s \]
Random Walk Centrality == Current Flow Btw. Centrality (see [5])

\[(D - A) \cdot V = s\]

Laplacian is not invertible, det = 0, because system of eq. is overdetermined \(\Rightarrow\) set one \(V_c = 0\) and measure voltages relative to \(v\). \(\Rightarrow\) remove the \(v\)-th row and column (since \(V_c = 0\)) \(\Rightarrow\) now invertible

\[V = (D_v - A_v)^{-1} \cdot s\]  \((\text{matrix inversion: } O(n^3))\)

let us now add a \(v\)-th row and column back into \((D_v - A_v)^{-1}\) with values all equal to zero.
The resulting matrix we will denote \(T\).

\[\rightarrow V_i^{(st)} = T_{is} - T_{it}\]

\(\rightarrow\) current flow at node \(i\): \(I_i^{(st)} = \frac{1}{2} \sum_j A_{ij} |V_i^{(st)} - V_j^{(st)}|\)

Random Walk Centrality == Current Flow Btw. Centrality (see [5])

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Random Walk Centrality (see [5])

\[ V = (D - A)^{-1} \mathbf{s} \]

Kirchhoff’s point law (current conservation): total current in/out of node is zero.

\[ \sum_{j} A_{ij} (V_j - V_i) = 0 \quad \sum_{j} A_{ij} V_j = 0 \]

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\[ V = (D - A)^{-1} \mathbf{s} \]

Current at node 
\[ I_r^0 = \sum_j A_{jr} (V_j - V_r) \]

unit current at nodes s and t
\[ I_s^0 = \sum_j A_{sj} (V_j - V_s) \]
\[ I_t^0 = \sum_j A_{tj} (V_j - V_t) \]

\(O((m+n)^2)\) for everything
\(O((m+n)^3)\) for everything
\(O((m+n)^4)\) for everything
\(O((m+n)^5)\) for everything

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- current flow at node $i$:
  \[
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  = \frac{1}{2} \sum_j A_{ij} |T_{is} - T_{it} - T_{js} + T_{jt}|, \quad \text{for } i \neq s, t.
  \]

- unit current flow at nodes $s$ and $t$:
  \[
  I^{(st)}_s = 1, \quad I^{(st)}_t = 1.
  \]

- cfb(i) (denoted as $b_i$) is then:
  \[
  b_i = \frac{1}{2} \sum_{s < t} I^{(st)}_i \\
  \quad \quad \text{(takes O(m n^2) for all i)} \rightarrow \text{(plus matrix inversion:)} \quad O((m+n) n^2) \text{ for everything}
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  \[
  b_i = \frac{1}{2} \sum_{s < t} I^{(st)}_i \\
  \quad \quad \text{(takes O(m n^2) for all i)} \rightarrow \text{(plus matrix inversion:)} \quad O((m+n) n^2) \text{ for everything}
  \]

Random Walk Centrality == Current Flow Btw. Centrality (see [5])

- random walk betweenness centrality (rwb):
  \[
  b_i = \text{random walk betweenness centrality (rwb)}
  \]

- rwb(i): move around „messages“: start (absorbing) random walk at $s$, end at $t$:
  \[
  \text{rwb(i):= net number of times that a message passes through } i \text{ on its journey (averaged over a large number of trials and averaged over s, t)}
  \]
  \[
  \text{„net“ number of times: „cancel back and fourth passes“}
  \]

- if in $i$, probability that in next step $j$:
  \[
  M_{ij} = \frac{A_{ij}}{k_j}, \quad \text{for } j \neq t,
  \]

\[
M = A \cdot D^{-1}
\]
with $D = \text{diag}(k_i)$
\[
D_{ii} = k_i
\]