Title: groh: profile1 (30.06.2015)
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Duration: 73:02 min
Pages: 83

Nash Equilibrium: More than one NE

- Another example: Battle of the sexes
  
  Two pure NE: (F;F) and (B;B)
  
  One mixed NE: Indifference condition
  \[ \sigma_1(x) = x \text{ and } \sigma_2(y) = y \]
  \[
  0 \cdot y + 2(1-y) = 1 \cdot y + 0 (1-y) \Rightarrow y=2/3
  
  Player 1’s indifference:
  \[
  0 \cdot x + 2(1-x) = 1 \cdot x + 0 (1-x) \Rightarrow x=2/3
  
  \Rightarrow \text{Mixed NE: } \{(2/3, 1/3); (2/3, 1/3)\}
  
- Another example: Game of chicken
  
  (same reasoning) \Rightarrow
  Mixed NE: \{(1/2, 1/2); (1/2, 1/2)\}

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Nash Equilibrium: More than one NE

- Another example: Battle of the sexes
  - Two pure NE: (F,F) and (B;B)
  - One mixed NE: Indifference condition
    → Let \( \sigma_i(F) = x \) and \( \sigma_i(B) = y \)
    → Player 1’s indifference:
      \[ 0 \cdot y + 2(1-y) = 1 \cdot y + 0(1-y) \rightarrow y = \frac{2}{3} \]
    → Player 2’s indifference:
      \[ 0 \cdot x + 2(1-x) = 1 \cdot x + 0(1-x) \rightarrow x = \frac{2}{3} \]
    → Mixed NE: \( \begin{pmatrix} (2/3, 1/3); (2/3, 1/3) \end{pmatrix} \)

- Another example: Game of chicken
  - (Same reasoning) → Mixed NE: \( \begin{pmatrix} (1/2, 1/2); (1/2, 1/2) \end{pmatrix} \)

Risk Dominance / Pareto Optimality

- In this game: (Among others) two pure NE: (U,L) and (D,R);
  (U,L) Pareto dominates (D,R)
- But: For player 1 D is safer (guarantees min payoff of 7) → if p(R) > 1/8 don’t go for (U,L) → no certainty!
- Pregame-communication / agreement on (U,L) ?!
  No: player 2 gains if player 1 plays U → player 2 will always tell „L” regardless of true intentions → agreement is worthless
### Nash Equilibrium: More than one NE

#### Risk Dominance / Pareto Optimality

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>U</strong></td>
<td>9,9</td>
<td>0,8</td>
</tr>
<tr>
<td><strong>D</strong></td>
<td>8,0</td>
<td>7,7</td>
</tr>
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- In this game: (Among others) two pure NE: \((U, L)\) and \((D, R)\); \((U, L)\): Pareto dominates \((D, R)\)
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Games in Strategic Form & Nash Equilibrium

Nash Equilibrium: More than one NE
Risk Dominance / Pareto Optimality

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<td>0,0,0</td>
<td>-5,-5,0</td>
</tr>
<tr>
<td>D</td>
<td>-5,-5,0</td>
<td>1,1,-5</td>
</tr>
</tbody>
</table>

- Three player game: Two pure NE: (U,L,A) and (D,R,B); (and one mixed); (U,L,A) pareto-dominates (D,R,B)
- If player 3’s choice is fixed → Two player game → (D,R) is pareto-dominant → if players 1 and 2 expect A: coordinate on (D,R).
- Concept of „coalition proof eq." (here (D,R,B)) (see [1])
Nash Equilibrium: More than one NE
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<tr>
<td></td>
<td>-5</td>
<td>1,</td>
</tr>
<tr>
<td></td>
<td>L</td>
<td>R</td>
</tr>
<tr>
<td>U</td>
<td>-2</td>
<td>-5</td>
</tr>
<tr>
<td>D</td>
<td>-5</td>
<td>1,</td>
</tr>
<tr>
<td></td>
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</tr>
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- If player 3’s choice is fixed → Two player game → (D,R) is pareto-dominant → if players 1 and 2 expect A: coordinate on (D,R).
- → concept of “coalition proof eq.” (here (D,R,B))(see [1])

Mixed Nash Equilibrium: General Analysis for 2 x 2 Games (see [2])

Player B

<table>
<thead>
<tr>
<th>q</th>
<th>1-q</th>
</tr>
</thead>
<tbody>
<tr>
<td>L</td>
<td>U</td>
</tr>
<tr>
<td></td>
<td>a_U</td>
</tr>
<tr>
<td></td>
<td>a_U</td>
</tr>
<tr>
<td>R</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
</tr>
</tbody>
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Player A

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>U</td>
<td>a_U</td>
</tr>
<tr>
<td></td>
<td>a_U</td>
</tr>
<tr>
<td>D</td>
<td></td>
</tr>
<tr>
<td>a_D</td>
<td></td>
</tr>
</tbody>
</table>

- Pure NE: One cell →
For A: cell’s payoff for A must be (weak) maximum over rows in that column
For B: cell’s payoff for B must be (weak) maximum over columns in that row
- Example: (U,R) is pure NE if \(a_{UR} \geq a_{DR}\) and \(b_{UR} \geq b_{UL}\)
Mixed Nash Equilibrium: General Analysis for 2 x 2 Games (see [2])

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<tbody>
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<td>u, aUL, bUL</td>
<td>aUL, bUL</td>
<td></td>
</tr>
<tr>
<td>1-p</td>
<td>d, aDL, bDL</td>
<td>aDL, bDL</td>
<td></td>
</tr>
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- Mixed NE: Start with player A's view
- If B plays mixed strategy (q, 1-q) or short q, A's best response is (p*(q), 1-p*(q)) or short p*(q).
- $u_A(p,q) = pq a_{UL} + p(1-q) a_{UR} + (1-p)q a_{DL} + (1-p)(1-q) a_{DR}$
- $u_B(p,q) = pq a_{UL} + p(1-q) a_{UR} + (1-p)q a_{DL} + (1-p)(1-q) a_{DR}$
- Now: case differentiation w.r.t. sign of $\delta(q)$: $u_A(p,q)$ will be maximal for:
  - $p^*=1$ if $\delta(q)$ is positive
  - $p^*=0$ if $\delta(q)$ is negative
  - all p in [0,1] if $\delta(q)$ is zero

Mixed Nash Equilibrium: General Analysis for 2 x 2 Games (see [2])

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Further case differentiation:

1. If $\delta(q)=0$ for all q $\Rightarrow$ for all p, A's best response are all p in [0,1]. ($\Rightarrow$ A is completely indifferent)
2. If $\delta(q)$ is affine linear: Has at most one zero q* with $\delta(q^*)=0$. (Exactly one if we set for q* $\Rightarrow$ for (a_{UL} - a_{UR} - a_{DL} + a_{DR}) = 0 $\Rightarrow$
3. If q* [0,1]: A plays p*1 or p*0 depending on sign of $\delta(q)$ ($\Rightarrow$ A has a dominant pure strategy)
4. If q* [0,1]: A plays p*1 or p*0 if $\delta(q)=0$ (depending on sign of $\delta(q)$) and can play any p if q=q* ($\Rightarrow$ A plays strategically)
Mixed Nash Equilibrium: General Analysis for 2 x 2 Games (see [2])

\[ u_A(p, q) = pq \delta(q) + (a_{DL} - a_{DR})q + a_{DR} \]

• Further case differentiation:

  • (1) If \( \delta(q) = 0 \) for all q \( \rightarrow \) for all q, A's best response are all p in [0,1]. \( \rightarrow \) A is completely indifferent
  • If \( \delta \) is not identically zero: \( \delta(q) \) is affine linear: Has at most one zero q' with \( \delta(q') = 0 \). (exactly one if we set for \( q' = \infty \) for \( (a_{UL} - a_{IR} - a_{DL} + a_{DR}) = 0 \) \( \rightarrow \) A has a dominant pure strategy
  • (2) If \( q' \epsilon [0, 1] \): A plays \( p^* = 1 \) or \( p^* = 0 \) depending on sign of \( \delta(q) \). \( \rightarrow \) A plays strategically
  • (3) If \( q' \epsilon [0, 1] \): A plays \( p^* = 1 \) or \( p^* = 0 \) if \( \delta(q) = 0 \) (depending on sign of \( \delta(q) \)) and can play any p if \( q = q^* \). \( \rightarrow \) A plays strategically

• Mixed NE: Start with player A's view

\[
\begin{array}{c|cc|c|cc}
\hline
& L & 1-p & \hline
q & p & a_{UL} & b_{UL} & a_{UR} & b_{UR} \\
\hline
1-p & a_{DL} & b_{DL} & a_{DR} & b_{DR} \\
\hline
\end{array}
\]

• If B plays mixed strategy \( (q, 1-q) \) or short q, A's best response is \( (p^*(q), 1-p^*(q)) \) or short p\(^*\)(q)

• \( u_A(p, q) = pq a_{UL} + p(1-q) a_{UR} + (1-p)q a_{DL} + (1-p)(1-q) a_{DR} \)

• Now: case differentiation w.r.t. sign of \( \delta(q) \): \( u_A(p, q) \) will be maximal for:
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Mixed Nash Equilibrium: General Analysis for 2 x 2 Games (see [2])

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<td>a_{ur}, b_{ur}</td>
</tr>
<tr>
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- If B plays mixed strategy (q, 1-q) or short q, A's best response is (p*(q), 1-p*(q)) or short p*(q).

\[ u_A(p,q) = p a_{ul} + (1-p)q a_{ur} + (1-p)(1-q) a_{dr} + p [ (a_{ul} - a_{dr}) q + (a_{ur} - a_{dl}) q + (a_{dl} - a_{ur}) q + a_{dr} ] + (a_{dl} - a_{ur}) q + a_{dr} \]

- Now: case differentiation w.r.t. sign of \( \delta(q) \): \( u_A(p,q) \) will be maximal for:
  - \( p* = 1 \) if \( \delta(q) \) is positive
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2. If \( \delta \) is not identically zero: \( \delta(q) \) is affine linear: Has at most one zero \( q* \) with \( \delta(q*)=0 \). (exactly one if we set for \( q^* = \infty \) for \( a_{ul} - a_{dr} \) \( a_{dl} + a_{ur} = 0 \))

3. If \( \delta \) is strictly increasing:
   - If \( q* \in [0,1] \): A plays \( p* = 1 \) or \( p* = 0 \) depending on sign of \( \delta(q) \). \( \Rightarrow \) A has a dominant pure strategy
   - If \( q* \in (0,1) \): A plays \( p* = 1 \) or \( p* = 0 \) depending on sign of \( \delta(q) \). \( \Rightarrow \) A plays strategically
Mixed Nash Equilibrium: General Analysis for 2 x 2 Games (see [2])

Vizualization of case (3)

(If q*=0 if 0<q<q* or if q=0 (depending on sign of δ(q)) and can play any p if q=q* (⇒ A plays strategically))

\[ u_A(p,q) = p \delta(q) + (a_{DL} - a_{DR})q + a_{DR} \]

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- Now: Analysis for B:
  - If A plays mixed strategy (p, 1-p) or short p, B’s best response is (q*(p), 1-q*(p)) or short q*(p).
  - \[ u_B(p,q) = q \{ (b_{UL} - b_{UR} - b_{DL} + b_{DR})p + b_{DL} - b_{DR} \} + (b_{UR} - b_{DR})p + b_{DR} \]
    \[ = q \delta(p) + (b_{UL} - b_{DR})p + b_{DR} \]
  - Exactly the same case differentiations as in case of A
  - **Trivially: Nash-Equilibrium iff q*(p) = p*(q).**

Mixed Nash Equilibrium: General Analysis for 2 x 2 Games (see [2])

| Player B |
|-------------------|---|
| q | 1-q |
| L | R |
| U | a_{UL}, b_{UL} |
| D | a_{DL}, b_{DL} |

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Multi-Stage Games with Observed Actions

- Preliminaries
  - In part1: Players made choices simultaneously
  - Now: order of choices makes knowledge explicit → Multi-Stage Games with Observed Actions
  - Example: Stackelberg equilibrium in Cournot model: Player1 chooses output level q₁ first, player 2 observes q₁ and then chooses q₂.
  - Assume costless production and linear demand → \[ u_i(q_1, q_2) = (12q_1 - q_2)q_2 \]
  - Player 2’s reaction on q₁ will be \[ s_2(q_1) \rightarrow u_i(q_1, q_2) = u_i(q_1, s_2(q_1)) \]
  - Pure NE: Neither player can gain by switching from NE to other pure strategy:
    - (1) For given q₁: Maximize: \[ max_{q_2} u_2(q_1, q_2) \rightarrow u_2(q^*_1, q^*_2) \]
      \[ 6 - q_2^* \rightarrow NE (Stackelberg EQ): (q^*_1, s^*_2(q^*_1)) \]
Nash Equilibrium: Example: Cournot Competition

- **Cournot model: Duopoly.** Each of two firms (players) $i$ produces same good.
- Output levels $q_i$ are chosen from sets $Q_i$
- Cost of production is $c_i(q_i)$
- Market price is $p(q) = p(q_1, q_2)$
- Firm $i$'s profit is then $u_i(q_1, q_2) = q_i p(q) - c_i(q_i)$
- Cournot reaction functions $r_1 : Q_2 \rightarrow Q_1$ and $r_2 : Q_1 \rightarrow Q_2$ specify optimal reaction on output level of opponent

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Nash Equilibrium: Example: Cournot Competition

- **Under certain reasonable assumptions (see [1]) we can maximize e.g.**
  \[ u_2(q_1, q_2) \] by solving \[ \frac{\partial}{\partial q_2} q_2 (q_1, q_2) = 0 \] which yields
  \[ \frac{\partial}{\partial q_2} (q_1, q_2) c_2(q_2) = p(q_1, q_2) + p'(q_1, q_2) q_2 c_2(q_2) = 0. \]
- **Inserting** \[ r_2(q_1) \] for \[ q_2 \]
  \[ p(q_1 + r_2(q_1)) + p'(q_1 + r_2(q_1)) r_2(q_1) - c_2'(r_2(q_1)) = 0 \]
  gives the defining equation for \[ r_2(\cdot) \].
  (analogous for \[ r_1(\cdot) \]).

- **The intersections** of the functions \[ r_2 \] and \[ r_1 \] are the NE of the Cournot game.

- **Example:** Linear demand \[ p(q) = \max(0, 1-q) \]; linear cost: \[ c_i(q_i) = c q_i \];
  \[ \rightarrow r_2(q_1) = 1/2 (1 - q_1 - c) \]; \[ r_1(q_2) = 1/2 (1 - q_2 - c) \];
  \[ \rightarrow \text{NE: } q^*_1 = r_2(q^*_1) = 1/3 (1-c) = q^*_1 = r_1(q^*_1) \]
Nash Equilibrium: Example: Cournot Competition

- Under certain reasonable assumptions (see [1]) we can maximize e.g. \(u_2(q_1, q_2)\) by solving \(\partial / \partial q_2 u_2(q_1, q_2) = 0\) which yields

\[
\frac{\partial}{\partial q_2} [q_2 p(q_1, q_2) - c_2(q_2)] = p(q_1, q_2) + p'(q_1, q_2) q_2 - c'_2(q_2) = 0.
\]

Inserting \(r_2(q_1)\) for \(q_2\)

\[
p(q_1 + r_2(q_1)) + p'(q_1 + r_2(q_1)) r_2(q_1) - c'_2(r_2(q_1)) = 0
\]
gives the defining equation for \(r_2(.)\).

(analogous for \(r_1(.)\)).

- The intersections of the functions \(r_2\) and \(r_1\) are the NE of the Cournot game.

**Example:** Linear demand \(p(q) = \max(0, 1-q)\); linear cost: \(c_i(q_i) = c q_i\):

\(\rightarrow r_2(q_1) = 1/2 (1 - q_1 - c); \quad r_1(q_1) = 1/2 (1 - q_2 - c);\)

\(\rightarrow \text{NE: } q^*_2 = r_2(q^*_1) = 1/3 (1 - c) = q^*_1 = r_1(q^*_1)\)

Nash Equilibrium: Example: Cournot Competition

- Under certain reasonable assumptions (see [1]) we can maximize e.g. \(u_2(q_1, q_2)\) by solving \(\partial / \partial q_2 u_2(q_1, q_2) = 0\) which yields

\[
\frac{\partial}{\partial q_2} [q_2 p(q_1, q_2) - c_2(q_2)] = p(q_1, q_2) + p'(q_1, q_2) q_2 - c'_2(q_2) = 0.
\]

Inserting \(r_2(q_1)\) for \(q_2\)

\[
p(q_1 + r_2(q_1)) + p'(q_1 + r_2(q_1)) r_2(q_1) - c'_2(r_2(q_1)) = 0
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gives the defining equation for \(r_2(.)\).

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Nash Equilibrium: Example: Cournot Competition

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  \[
  \frac{\partial}{\partial q_2} \left[ p(q_1, q_2) - c(q_2) \right] = p'(q_1, q_2) q_2 - c'(q_2) = 0.
  \]

  Inserting \( r_2(q_1) \) for \( q_2 \)
  \[
  p(q_1, + r_2(q_1)) + p'(q_1, r_2(q_1)) r_2(q_1) - c'(r_2(q_1)) = 0
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gives the defining equation for \( r_2(.) \).

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  The intersections of the functions \( r_2 \) and \( r_1 \) are the NE of the Cournot game.

  **Example**: Linear demand \( p(q) = \max(0, 1-q) \); linear cost: \( c(q_i) = c q_i \):
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  r_2(q_1) = \frac{1}{2} (1-q_1 - c); \quad r_1(q_2) = \frac{1}{2} (1-q_2 - c);
  \]
  \[
  \Rightarrow \text{NE:} \quad q^*_2 = r_2(q^*_2) = \frac{1}{3} (1-c) = q^*_1 = r_1(q^*_2)
  \]

Multi-Stage Games with Observed Actions

In part1: Players made choices simultaneously

Now: order of choices makes knowledge explicit → Multi-Stage Games with Observed Actions

**Example**: Stackelberg equilibrium in Cournot model: Player1 chooses output level \( q_1 \), first, player 2 observes \( q_1 \) and then chooses \( q_2 \).

Assume costless production and linear demand

\[
 u_i(q_i, q_j) = (12 - q_j) q_i
 \]

Player 2's reaction on \( q_1 \) will be \( s_2(q_1) \) → \( u_i(q_1, q_2) = u_i(q_1, s_2(q_1)) \)

Pure NE: Neither player can gain by switching from NE to other pure strategy:

1. For given \( q_1 \): Maximize: \( \max_{q_2} u_2(q_1, q_2) \) → we get Cournot reaction \( r_2(q_1) = \frac{6}{q_2} \); Now: Maximize: \( \max_{q_1} u_1(q_1, r_2(q_1)) \) → \( q^*_1 = 6 \) → NE (Stackelberg EQ): \( (q^*_1, r_2(q^*_1)) \)
Multi-Stage Games with Observed Actions

Preliminaries

- In part 1: Players made choices **simultaneously**
- Now: order of choices makes knowledge explicit $\rightarrow$ Multi-Stage Games with Observed Actions

- **Example:** Stackelberg equilibrium in Cournot model: Player 1 chooses output level $q_1$ first, player 2 observes $q_1$ and then chooses $q_2$.
- Assume costless production and linear demand $\rightarrow$ $u_1(q_1, q_2) = [12 - q_1 - q_2] q_1$
- Player 2's reaction on $q_1$ will be $s_2(q_1) \rightarrow u_1(q_1, q_2) = u_1(q_1, s_2(q_1))$
- Pure NE: Neither player can gain by switching from NE to other pure strategy:
  - (1) For given $q_1$: Maximize: $\max_{q_2} u_1(q_1, q_2) \rightarrow$ we get Cournot reaction $r_1(q_1) = 6 - q_1/2$; Now: Maximize: $\max_{q_1} u_1(q_1, r_1(q_1)) \rightarrow q^*_1 = 6 \rightarrow$ NE (Stackelberg EQ): $(q^*_1, r_1(q^*_1))$

- **Other NE:** Cournot Equilibrium (from simultaneous play)
  $(q^*_1, q^*_2) = (4; 4)$

- **Why is this a NE?** If player 2 plays $q^*_2$, player 1 must play $r_1(q^*_2) = q^*_1$ in order to maximize $u_1$. If player 1 plays $q^*_1$, player 2 can play any $q_2$ with $s_2(q^*_1) = q^*_2$ in order to maximize $u_2$, including $q_2 = \text{const} = q^*_2$.

- Stackelberg EQ $(q^*_1, r_1(q^*_1))$: payoffs is $(18; 9)$;
  Cournot EQ $(q^*_1, q^*_2)$: payoffs is $(16; 16)$

- But: player 2 plays $q^*_2$ (independent of $q_1$) is empty threat: If player 1 plays $q_1 = q^*_1$, player 2 is better off with $r_2(q_1)$

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Multistage Games with Observed Actions

Preliminaries

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  Cournot EQ \( (q^*_1; q^*_2) \): payoff is \( 16; 16 \);

- **But**: player 2 plays \( q^*_2 \) (independent of \( q_1 \)) is empty threat: If player 1 plays \( q_1 \neq q^*_1 \), player 2 is better of with \( r_2(q_1) \).

Multistage Games

- **Stages \( 0, \ldots, K \)**: At each stage \( k \): all players choose actions simultaneously; All players know actions from all players from all previous stages.

- Action profile of stage \( k \): \( a^k = (a^k_1, a^k_2, \ldots, a^k_k) \);

- History at end of stage \( k \): beginning of stage \( k+1 \): 
  \( h^{k+1} = (a^k, a^{k-1}, \ldots, a^1) \); 
  \( h^0 = \emptyset \)

- Set of possible actions of player \( i \) if history \( h^k \): \( A_i(h^k) \)

- Possible: \( A_i(h^k) = \{ \text{do nothing} \} \) → Previous example and generally games of „alternating choice“ included.
Multi-Stage Games

- **Stages \((0,\ldots,K)\);** At each stage \(k\): all players choose actions simultaneously; All players know actions from all players from all previous stages
  - **Action profile of stage \(k\):** \(a_k^k = (a_{1k}^k, a_{2k}^k, \ldots, a_{nk}^k)\)
  - **History at end of stage \(k\) / beginning of stage \(k+1\):** \(h_{k+1}^k = (a_0^k, a_1^k, \ldots, a_{k}^k); h_0^k = \emptyset\)
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Multi-Stage Games

- **Pure strategy** for player \(i\) is contingent plan \(s_i = (s_i^0, s_i^1, \ldots, s_i^K)\) where the \(s_i^k\) map the history to set of possible actions at each stage: \(s_i^k(h^k) \subseteq A_i(h^k)\)
- **Mixed strategy** \(\Omega_i(h^k):\) define prob.-distribution over \(A_i(h^k)\)
- Thus: \(a_i^k = s_i^k(h^k) = s_i^k(a_0^k, a_1^k, \ldots, a_{k-1}^k)\)
- \((a_0^k, a_2^k, \ldots, a_i^k)\) is called **path**
- **Overall Payoff of player \(i\):** \(u_i(h_{k+1}^k)\)

**Pure strategy NE:** Strategy profile \(s\) so that for all players \(i:\)
\[ u_i(s_i, s_i) \geq u_i(s_i, s_i) \quad \text{for all } s_i \]
**Multi-Stage Games**

- **Pure strategy** for player $i$ is contingent plan $s_i = (s_i^0, s_i^1, ..., s_i^k)$ where the $s_i^k$ map the history to set of possible actions at each stage: $s_i^k(h^k) \in A_i(h^k)$;
- **Mixed strategy** $\Theta_i^k(h^k)$: define prob.-distribution over $A_i(h^k)$
- **Path** $\left( a^0, a^1, ..., a^k \right)$ is called path
- **Overall Payoff** of player $i$: $u_i(h^{k+1})$

Pure strategy NE: Strategy profile $s$ so that for all players $i$, $u_i(s_i, s_{-i}) \geq u_i(s'_{i}, s_{-i})$ for all $s'_i$

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**Backward Induction & Subgame Perfection**

- Stackelberg EQ example: Alternating moves: backward induction easy:
- Multistage game has **perfect information** if at each stage, only one player has a non-trivial action set. (alternating moves)
- We regard finite games although there is a generalization of backward induction to infinite games
- Observation: If games get long: “Rationality” of backward induction (player moving at stage 0 must forecast that player at stage 1 will forecast that player at stage 2....that player at stage K-1 will forecast the move of player at stage K”) becomes weaker with growing K.
Backward Induction & Subgame Perfection

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Subgame perfection: Example: Two firms, two stages:
First stage: firm 1 chooses to invest in technology or not;
Second stage: firm 1 and 2 choose output levels q1 and q2 simultaneously
- stage 2 simultaneous choice → Not perfect information
- But: Each history h1 generates normal "strategic form-like" subgame → compute NE → subgame-perfect NE
- Stage k: History before k is h^k → Game from stage k on is G(h^k)

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Multi-Stage Games with Observed Actions

Backward Induction & Subgame Perfection

- **Subgame perfection: Example:** Two firms, two stages:
  - **First stage:** firm 1 chooses to invest in technology or not;
  - **Second stage:** firm 1 and 2 choose output levels $q_1$ and $q_2$ simultaneously
- **Stage 2 simultaneous choice → Not perfect information**
- **But:** Each history $h^k$ generates “normal” “strategic form-like” subgame $\rightarrow$ compute NE $\rightarrow$ subgame-perfect NE
- **Stage k:** History before k is $h^k$ $\rightarrow$ Game from stage k on is $G(h^k)$

Multi-Stage Games

- **Stages (0,...,K):** At each stage k: all players choose actions simultaneously; All players know actions from all players from all previous stages
- **Action profile of stage k:** $a^k = (a^k_1, a^k_2, ..., a^k_n)$
- **History at end of stage k / beginning of stage k+1:**
  $h^{k+1} = (a^0, a^1, ..., a^k)$; $h^0 = \emptyset$
- **Set of possible actions of player i if history $h^k$:** $A_i(h^k)$
- **Possible:** $A_i(h^k) = \{\text{do nothing}\}$ $\rightarrow$ Previous example and generally games of “alternating choice” included
Multi-Stage Games with Observed Actions

Backward Induction & Subgame Perfection

- Payoff in $G(h^k)$ is $u_i(h^{k+1})$ where $h^{k+1} = (h^k, a^{k+1}, a^{k+2}, \ldots, a^k)$
- Strategies in $G(h^k)$: As before: maps from histories to actions, where histories are consistent with $h^k$
  - We can define NE in $G(h^k)$ as before.
- Strategy profile $s$ (whole game) induces strategy profile $s | h^k$ on $G(h^k)$:
  For all players $s_i | h^k$ is restriction of $s_i$ to histories consistent with $h^k$

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• Strategies in $G(h^k)$: As before: maps from histories to actions, where histories are consistent with $h^k$

→ we can define NE in $G(h^k)$ as before.

• Strategy profile $s$ (whole game) induces strategy profile $s|_{h^k}$ on $G(h^k)$: For all players $s_i|_{h^k}$ is restriction of $s_i$ to histories consistent with $h^k$

Example from above: Two firms; two stages; cost per unit: 2; stage 1: firm1 can invest in new technology; cost $f$ which reduces cost per unit to 0; linear demand: $p(q) = 14-q$

• Firm 1 does not invest: payoff is $[14-q_1 \cdot q_2]q_1 - 2q_1 = [12-q_1 \cdot q_2]q_1$

• Firm 1 does invest: payoff is $[14-q_1 \cdot q_2]q_1 \cdot f$

• Firm 2: payoff is $[12-q_1 \cdot q_2]q_2$

Strategy profile $s$ (of whole multistage game with observed actions) is in subgame-perfect EQ if for all $h^k$ the restriction $s|_{h^k}$ on $G(h^k)$ is a NE of $G(h^k)$

• In games of perfect information ("alternating"): This definition reduces to backward induction

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**Example from above:** Two firms; two stages; cost per unit: 2; stage 1: firm 1 can invest in new technology: cost \( f \) which reduces cost per unit to 0; linear demand: \( p(q) = 14-q \)
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- Firm 1 does invest: payoff is \([14-q_1-q_2]q_1 - f \)
- Firm 2: payoff is \([12-q_1-q_2]q_2 \)

**Firm 1 does not invest:** reaction functions of firms are \( r_1(q_1) = 6 - q_1 / 2 \) → Cournot EQ: \((q_1^*; q_2^*)=(4; 4)\): payoff is \((16; 16)\);

**Firm 1 does invest:** reaction function of firm 1 changes to \( r_1^-(q_1) = 7 - q_1 / 2 \) → NE changes to: \((16; 10; 3)\):
- firm 1’s payoff is \( 256/9 - f \);
- \( \rightarrow \) Invest if \( 256/9 - f > 16 \)
- Investing changes firm 1’s payoff in two ways:
  1. lower production cost
  2. \( q_2 \) goes down because firm 1 can be more aggressive \( (r_1^-(q_2) > r_1(q_2)) \) for all \( q_2 \)