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### More Notation:

- Discussing player i’s strategy-options, holding other player’s options fixed:
  
- \( s_i \in S_i \) „other player’s strategies“

- Short notation: \((s'_1, s'_2) := (s_1, \ldots, s_{i-1}, s'_i, s_{i+1}, \ldots, s_n)\)

- Same for mixed strategies: \((\sigma'_1, \sigma'_2) := (\sigma_1, \ldots, \sigma_{i-1}, \sigma'_i, \sigma_{i+1}, \ldots, \sigma_n)\)

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### Definition:

- Pure strategy \( s_i \) is **strictly dominated** for player i if \( \sigma'_i \) exists so that \( u_i(\sigma'_i, s_{-i}) > u_i(s_i, s_{-i}) \) for all \( s_{-i} \in S_{-i} \)

- ... weakly dominated:
  
  \( u_i(\sigma'_i, s_{-i}) \geq u_i(s_i, s_{-i}) \) for all \( s_{-i} \in S_{-i} \) (and \( > \) for at least one \( s_{-i} \))

- If \( u_i(\sigma'_i, s_{-i}) > u_i(s_i, s_{-i}) \) for all \( s_{-i} \in S_{-i} \), we also have

  \( u_i(\sigma'_i, \sigma_{-i}) > u_i(s_i, \sigma_{-i}) \) for all \( \sigma_{-i} \in S_{-i} \), because \( u_i(\sigma'_i, \sigma_{-i}) \) is a convex function of \( u_i(\sigma'_i, s_{-i}), u_i(s'_i, s_{-i}), u_i(\sigma'_i, s'_{-i}), \ldots \).

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### Notation: Strategic Form Games

- **Two Player zero sum game:**

  \[ \forall s : \sum_{i=1}^{2} u_i(s) = 0 \]

- Structure of game is common knowledge:
  
  all players know;  
  all players know that all players know;  
  all players know that all players know that all players know;  
  ...

- **Mixed strategy** \( \sigma_i : S_i \rightarrow [0,1] \):
  
  Probability distribution over pure strategies (statistically independent for each player);

  **Examples:** \( \sigma_i(U) = 1/3, \ \sigma_i(M) = 2/3, \ \sigma_i(D) = 0; \)

  \( \sigma'_i(U) = 2/3, \ \sigma'_i(M) = 1/6, \ \sigma'_i(D) = 1/6; \)

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  Thus: \( \sigma_i(s_i) \) is the probability that player i assigns to strategy (action) \( s_i \)

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Two Player zero sum game:
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Mixed strategy \( \sigma_i : S_i \rightarrow [0,1] \) Probability distribution over pure strategies (statistically independent for each player);
Examples:
\( \sigma_1(U) = 1/3, \sigma_1(M) = 2/3, \sigma_1(D) = 0 \);
\( \sigma_1'(U) = 2/3, \sigma_1'(M) = 1/6, \sigma_1'(D) = 1/6 \);

Thus: \( \sigma_i(s_j) \) is the probability that player \( i \) assigns to strategy (action) \( s_j \).

Example:
Let
\[ \sigma_1(U) = 1/3, \quad \sigma_1(M) = 1/3, \quad \sigma_1(D) = 1/3 \]
\[ \sigma_2(L) = 0, \quad \sigma_2(M) = 1/2, \quad \sigma_2(R) = 1/2 \]
or short
\[ \sigma_1 = (1/3, 1/3, 1/3) \]
\[ \sigma_2 = (0, 1/2, 1/2) \]

We then have:
\[ u_1(\sigma_1, \sigma_2) = 1/3 (0*4 + 1/2*5 + 1/2*6) + 1/3 (0*2 + 1/2*8 + 1/2*3) + 1/3 (0*3 + 1/2*9 + 1/2*2) = 11/2 \]
\[ u_2(\sigma_1, \sigma_2) = ... = 27/6 \]
Games in Strategic Form & Nash Equilibrium

What is rational to do?

- No matter what player 1 does: R gives player 2 a strictly higher payoff than M.
- "M is strictly dominated by R" 
- \( \rightarrow \) player 1 knows that player 2 will not play M \( \rightarrow U \) is better than M or D

\[ \begin{array}{c|ccc}
 & L & M & R \\
\hline
U & 4, 3 & 5, 1 & 6, 2 \\
M & 2, 1 & 8, 4 & 3, 6 \\
D & 3, 0 & 9, 6 & 2, 8 \\
\end{array} \]

- \( \rightarrow \) player 2 knows that player 1 knows that player 2 will not play M \( \rightarrow \) player 2 knows that player 1 will play U \( \rightarrow \) player 2 will play L

This elimination process: "iterated strict dominance"

Is outcome dependent on elimination order?

No! If \( s_i \) is strictly worse than \( s_i' \) against opponent’s strategy in set D then \( s_i \) is strictly worse than \( s_i' \) against opponent’s strategy in any subset of D

Games in Strategic Form & Nash Equilibrium

More Notation:

- Discussing player i’s strategy-options, holding other player’s options fixed:
  - \( s_i \in S_i \): “other player’s strategies”
  - Short notation: \( (s_i', s_{-i}) := (s_1, \ldots, s_{i-1}, s_i', s_{i+1}, \ldots, s_n) \)
  - Same for mixed strategies: \( (\sigma_i', \sigma_{-i}) := (\sigma_1, \ldots, \sigma_{i-1}, \sigma_i', \sigma_{i+1}, \ldots, \sigma_n) \)

Definition:

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  - If \( u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}) \) for all \( s_i \in S_i \) we also have
  - \( u_i(s_i', \sigma_{-i}) > u_i(s_i, \sigma_{-i}) \) for all \( \sigma_{-i} \in S_{-i} \) because
  - \( u_i(s_i', \sigma_{-i}) \) is a convex function of \( u_i(s_i', s_{-i}), u_i(s_i', s_{-i}', \ldots), u_i(s_i', s_{-i'', \ldots}) \)....

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Games in Strategic Form & Nash Equilibrium

What about dominated mixed strategies?

- Easy: A mixed strategy that assigns positive probabilities to pure strategies that are dominated is dominated
- But: A mixed strategy may be dominated even if it assigns positive probabilities to pure strategies that are not even weakly dominated:

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Example:

- U and M are not dominated by D for player 1
- But: Playing \( \sigma_1 = (\frac{1}{2}, \frac{1}{2}, 0) \) gives expected utility \( u_1(\sigma_1, \sigma_{-1}) = 1/2 \) no matter what 2 plays \( \rightarrow D (\sigma_0 = (0, 0, 1)) \) dominates \( \sigma_1 \)
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  \( D(\sigma_0=(0, 0, 1)) \) dominates \( \sigma_1 \)

Games in Strategic Form & Nash Equilibrium

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A note on rationality

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- BUT: psychology $\Rightarrow$ play D instead of U because "U is unsafe"

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Game Theory ↔ Decision Theory

**Example**

- Iterated strict dominance $\rightarrow (U,L)$

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- If player 1 reduces his payoff for U by 2:
  - Decision theory: no use
  - Game theory: new iterated strict dominance $\rightarrow (D,R)$

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Prisoner’s dilemma & Iterated dominance

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- Iterated strict dominance $\rightarrow (D,D)$
Vickrey Auction & Iterated dominance

- **Good’s valuations:** $v_i$; Assume common knowledge for the moment
- **Bids:** $s_i$

**Second price:**
- winning condition: $s_i > \max_{j \neq i} s_j$
- let $r_i := \max_{j \neq i} s_j$; $r_i$ is the price having to be paid
- winner $i$’s utility: $u_i = v_i - r_i$; other players utility = 0

for each player bidding true valuation is weakly dominant:
- case $s_i > v_i$: (overbidding)
  - If $r_i > s_i$: loses $\Rightarrow u_i = 0$
    $\Rightarrow$ could have bidden $v_i$ as well
  - If $r_i \leq v_i$: wins $\Rightarrow u_i = v_i - r_i$
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Games in Strategic Form & Nash Equilibrium

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Games in Strategic Form & Nash Equilibrium

Vickrey Auction & Iterated dominance

- case $v_i < r_1 < s_i$
  - i wins $\Rightarrow u_i = v_i - r_1 < 0$ (winner’s curse)
  - should have bid $v_i = r_1 \Rightarrow u_i = 0$ at least

- case $s_i < v_i$ (underbidding)
  - If $r_1 \leq s_i$ or $r_1 \geq v_i$:
    - $u_i$ is unchanged if he bids $v_i$ instead of $s_i$
  - If $s_i < r_1 < v_i$:
    - bidder forgoes positive winning chances by underbidding

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Nash Equilibrium

- **Nash Equilibrium**: strategy profile: each player’s strategy is optimal response to all other player’s strategies:

- Mixed strategy profile \( \sigma^* \) is Nash Equilibrium if for all \( i \): \( u_i(\sigma^*_{-i}, \sigma^*_{+i}) \geq u_i(s_i, \sigma^*_{-i}) \) for all \( s_i \in S_i \).

  (Pure strategy profiles also possible \( \rightarrow \) “pure strategy NE”)

- Strategy profile \( s^* \) is **Strict Nash Equilibrium**: if it is a NE and for all \( i \): \( u_i(s^*_{-i}, s^*_{+i}) > u_i(s_i, s^*_{+i}) \) for all \( s_i \neq s^*_i \).

  Strict NE is necessarily a pure strategy NE by definition.
Nash Equilibrium

- From previous slide: \( \sigma^* \) is Nash Equilibrium if for all \( i \): \( u_i(\sigma^*_i, \sigma^*_{-i}) \geq u_i(\sigma_i, \sigma^*_{-i}) \) for all \( \sigma_i \in S_i \)
- Expected utilities are "linear in the probabilities"
  - In NE def we must only check for pure alternatives \( \sigma_i \)
  - In a (non-degenerate) mixed strategy Nash Equilibrium a player must be (a priori) indifferent between all pure strategies to which he assigns positive probability (Indifference condition)
    (we will analyze this in more depth later)

Indifference condition: more detailed explanation:

For player \( i \)'s utility, we have:

\[
u_i(\sigma) = \sum_{s_i \in S_u} \sigma_i(s_i) u_i(s_i, \sigma_{-i}) \quad \text{with} \quad \sum_{s_i \in S_u} \sigma_i(s_i) = 1
\]

for the NE \( \sigma^* \) we thus have:

\[
u_i(\sigma^*) = \sum_{s_i \in S_u} \sigma_i^*(s_i) u_i(s_i, \sigma_{-i}^*) \quad \text{with} \quad \sum_{s_i \in S_u} \sigma_i^*(s_i) = 1
\]

since \( u_i(\sigma^*) \) is the best outcome \( i \) can achieve, when the others play \( \sigma_{-i}^* \), all the \( u_i(s_i, \sigma_{-i}^*) \) with \( \sigma_i(s_i) > 0 \) must be equal, and equal to \( u_i(\sigma^*) \).

why? \( \rightarrow \) no \( u_i(s_i, \sigma_{-i}^*) \) can be greater than \( u_i(\sigma^*) \) otherwise the NE condition would be violated, and also not smaller, because then the sum would also be smaller.
Inference condition: more detailed explanation:

For player $i$’s utility, we have:

$$u_i(\sigma) = \sum_{s_i \in S_i} \sigma_i(s_i)u_i(s_i, s_{-i}) \quad \text{with} \quad \sum_{s_i \in S_i} \sigma_i(s_i) = 1$$

for the NE $\sigma^*$ we thus have:

$$u_i(\sigma^*) = \sum_{s_i \in S_i} \sigma^*_i(s_i)u_i(s_i, \sigma^*_{-i}) \quad \text{with} \quad \sum_{s_i \in S_i} \sigma^*_i(s_i) = 1$$

since $u_i(\sigma^*)$ is the best outcome $i$ can achieve, when the others play $\sigma^*_{-i}$, all the $u_i(s_i, \sigma^*_{-i})$ with $\sigma_i(s_i) > 0$ must be equal, and equal to $u_i(\sigma^*)$.

why? no $u_i(s_i, \sigma^*_{-i})$ can be greater than $u_i(\sigma^*)$ otherwise the NE condition would be violated, and also not smaller, because then the sum would also be smaller.

Notation: Strategic Form Games

- Space of mixed strategies for player $i$: $\Sigma_i$
- Space of mixed strategy profiles: $\Sigma = \times_i \Sigma_i$
- Mixed strategy profile $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_t) \in \Sigma$
- Player $i$’s payoff when a mixed strategy profile $\sigma$ is played is
  $$\sum_{s \in \Sigma} \left( \prod_{j=1}^{t} \sigma_j(s_j) \right) u_i(s)$$
  denoted as $u_i(\sigma)$, is a linear function of the $\sigma_i$
- A pure strategy of a player is a special mixed strategy of that player with one probability equal to 1 and all others equal to 0

Games in Strategic Form & Nash Equilibrium

Nash Equilibrium

- Nash Equilibrium: strategy profile: each player’s strategy is optimal response to all other player’s strategies:
  - Mixed strategy profile $\sigma^*$ is Nash Equilibrium if
    
  for all $i$: $u_i(\sigma^*_i, \sigma^*_{-i}) \geq u_i(s_i, \sigma^*_{-i})$ for all $s_i \in S_i$
  (Pure strategy profiles also possible → "pure strategy NE")
- Strategy profile $s^*$ is Strict Nash Equilibrium: if it is a NE and for all $i$: $u_i(s^*_i, s^*_{-i}) > u_i(s_i, s^*_{-i})$ for all $s_i \neq s^*_i$.
  Strict NE is necessarily a pure strategy NE by definition.

Games in Strategic Form & Nash Equilibrium

- What is rational to do?
  - No matter what player 1 does, $R$ gives player 2 a strictly higher payoff than $M$.
    - "M is strictly dominated by $R"$
  - $\rightarrow$ player 1 knows that player 2 will not play $M \rightarrow U$ is better than $M$ or $D$
  - $\rightarrow$ player 2 knows that player 1 knows that player 2 will not play $M \rightarrow$ player 2 knows that player 1 will play $U \rightarrow$ player 2 will play $L$
  - This elimination process: "iterated strict dominance"
  - Is outcome dependent on elimination order?
    - No! If $s_i$ is strictly worse than $s'_i$ against opponent’s strategy in set $D$ then $s_i$ is strictly worse than $s'_i$ against opponent’s strategy in any subset of $D"
Inference condition: more detailed explanation:

For player i’s utility, we have:

\[ u_i(\sigma) = \sum_{s_i \in S_u} \sigma_i(s_i) u_i(s_i, \sigma_{-i}) \quad \text{with} \quad \sum_{s_i \in S_u} \sigma_i(s_i) = 1 \]

for the NE \( \sigma^* \) we thus have:

\[ u_i(\sigma^*) = \sum_{s_i \in S_u} \sigma_i^*(s_i) u_i(s_i, \sigma^*_{-i}) \quad \text{with} \quad \sum_{s_i \in S_u} \sigma_i^*(s_i) = 1 \]

since \( u_i(\sigma^*) \) is the best outcome, i can achieve, when the others play \( \sigma^*_{-i} \), all the \( u_i(s_i, \sigma^*_{-i}) \) with \( \sigma_i(s_i) > 0 \) must be equal, and equal to \( u_i(\sigma^*) \).

why? \( \rightarrow \) no \( u_i(s_i, \sigma^*_{-i}) \) can be greater than \( u_i(\sigma^*) \) otherwise the NE condition would be violated, and also not smaller, because then the sum would also be smaller.

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why? $\Rightarrow$ no $u_i(s_i, \sigma^*_{-i})$ can be greater than $u_i(\sigma^*)$ otherwise the NE condition would be violated, and also not smaller, because then the sum would also be smaller.

---

Nash Equilibrium

- Strict equilibria need not exist. However each finite strategy form game has a mixed strategy equilibrium.
- In NE no player has incentive to deviate from NE
- In reality: If rationality is „non-strict“ (mistakes are made): deviations can occur
- If one round of elimination of strictly dominated strategies yields unique strategy profile, this strategy profile is a strict NE (unique)
- In NE, positive probabilities may only be assigned to not-strictly dominated strategies (Otherwise profit may be increased by choosing a dominating strategy).
Nash Equilibrium: Example: Cournot Competition

- **Cournot model: Duopoly.** Each of two firms (players) produces same good.
- Output levels \( q_i \) are chosen from sets \( Q_i \)
- Cost of production is \( c_i(q_i) \)
- Market price is \( p(q) = p(q_1 + q_2) \)
- Firm i’s profit is then \( u_i(q_1, q_2) = q_i p(q) - c_i(q_i) \)
- Cournot reaction functions \( r_1 : Q_2 \rightarrow Q_1 \) and \( r_2 : Q_1 \rightarrow Q_2 \) specify optimal reaction on output level of opponent

---

Under certain reasonable assumptions (see [1]) we can maximize e.g. \( u_2(q_1, q_1) \) by solving \( d/dq_2 \ u_2(q_1, q_2) = 0 \) which yields

\[
\frac{d}{dq_2} \ [q_2 \ p(q_1, q_2) - c_2(q_2)] = p(q_1, q_2) + p'(q_1, q_2) \ q_2 - c_2'(q_2) = 0.
\]

Inserting \( r_2(q_1) \) for \( q_2 \)

\[
p(q_1 + r_2(q_1)) + p'(q_1 + r_2(q_1)) \ r_2(q_1) - c_2'(r_2(q_1)) = 0
\]

gives the defining equation for \( r_2(.) \).

(analogous for \( r_1(.) \)).

The intersections of the functions \( r_2 \) and \( r_1 \) are the NE of the Cournot game.

Example: Linear demand \( p(q) = \max(0, 1-q) \); linear cost: \( c_i(q_i) = q_i \)

\[
\rightarrow r_2(q_1) = 1/2 \ (1 - q_1 - c);
\]

\[
r_1(q_2) = 1/2 \ (1 - q_2 - c);
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\[
\rightarrow \text{NE: } q^*_1 = r_1(q^*_2) = 1/3 \ (1 - c) = q^*_1 = r_1(q^*_2)
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Games in Strategic Form & Nash Equilibrium

Nash Equilibrium: Example: Cournot Competition

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- Example: Linear demand \( p(q) = \max(0, 1-q) \); linear cost: \( c_i(q_i) = c \ q_i \):

\[
\rightarrow r_2(q_1) = 1/2 (1- q_1 - c) ; \ r_1(q_2) = 1/2 (1- q_2 - c);
\]
\[
\rightarrow \text{NE: } q^*_2 = r_2(q^*_1) = 1/3 (1-c) = q^*_1 = r_1(q^*_1)
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Nash Equilibrium: Example: Cournot Competition

- Under certain reasonable assumptions (see [1]) we can maximize e.g. \( u_2(q_1, q_2) \) by solving \( d/dq_2 \ u_2(q_1, q_2) = 0 \) which yields

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Inserting \( r_2(q_1) \) for \( q_2 \)

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p(q_1 + r_2(q_1)) + p'(q_1 + r_2(q_1)) \ r_2(q_1) - c_2'(r_2(q_1)) = 0
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\[ r_2(q_1) = 1/2 (1 - q_1 - c) \]
\[ r_1(q_2) = 1/2 (1 - q_2 - c) \]

\( \implies \) NE: \( q^*_2 = r_2(q^*_1) = 1/3 (1-c) = q^*_1 = r_1(q^*_1) \)
Nash Equilibrium: Example: Hotelling Competition

- Two firms, 1 (at x=0) and 2 (at x=1) sell same good
- Unit cost of product := c; price for product of firm i := p_i
- Customers: uniformly distributed over [0,1] with probability density 1
- Customer transportation cost: t per length unit
- Customers: have unit demand; buy good if price + transportation_cost < max_price = S, buy good from overall cheaper firm

\[
D_1(p_1, p_2) = (p_2 - p_1 + t) / (2t)
\]

\[
D_2(p_2, r_2) = 1 - D_1(p_1, p_2)
\]

Nash Equilibrium \( (p_1^*, p_2^*) \): For each i: \( p_i^* \in \text{argmax} \{ (p_i - c) D_i(p_i, p_i^*) \} \)

Denoting the reaction functions by \( r_i(p_i) \) and \( r_2(p_1) \) we get for e.g. firm 2:

\[
\frac{d}{dp_2} \{ (p_2 - c) D_2(p_2, p_2) \} = 0 \quad \text{and afterwards insert} \quad r_2(p_1) \quad \text{for} \quad p_2 \rightarrow
\]

\[
D_2(p_2, r_2(p_2)) + (r_2(p_1) - c) \frac{d}{dp_2} D_2(p_2, r_2(p_1)) = 0
\]

\[
p_1^* = p_2^* = c + t \quad \text{for} \quad c + 3/2 t \leq S
\]
Nash Equilibrium: Example: Hotelling Competition

- Demand for firm 1 is \( D_1(p_1, p_2) = x \) where \( p_1 + tx = p_2 + t(1-x) \)
- \( \rightarrow D_1(p_1, p_2) = (p_2 - p_1 + t) / (2t) \)
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- Nash Equilibrium \( (p^*_1, p^*_2) \): For each \( i \): \( p^*_i \in \text{argmax} \{(p_i - c) D_i(p_i, p^*_i)\} \)
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  \[ \frac{d}{dp_2} \left\{ (p_2 - c) D_2(p^*_1, p_2) \right\} = 0 \quad + \quad \text{afterwards insert } r_2(p_1) \text{ for } p_2 \]
  \[ D_2(p_2, r_2(p_1)) + (r_2(p_1) - c) \frac{\partial}{\partial p_2} D_2(p_2, r_2(p_1)) = 0 \quad \rightarrow \]
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  - buy good from overall cheaper firm

Nash Equilibrium: Non-Existence-of Pure NE-Example

- Some games may have more than one pure strategy NE
- Not all games have a pure strategy NE:
  - Example: Matching pennies:
    - Both players simultaneously announce
      - Head or Tails: IF match \rightarrow 1 wins; If differ \rightarrow 2 wins
      - No pure NE;
        - mixed strategy NE: \left(\frac{1}{2}, \frac{1}{2}\right)
  - Reasoning: If player 2 plays \left(\frac{1}{2}, \frac{1}{2}\right) then player 1’s expected payoff is \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (-1) = 0 when playing H and \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0 when playing T \rightarrow player 1 is also indifferent

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- Example: Matching pennies:
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  - No pure NE; but mixed strategy NE: \( (1/2, 1/2); (1/2, 1/2) \):
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Nash Equilibrium: Non-Existence of Pure NE-Example 2

- Another example: Inspection game
- Worker: work or shirk; Employer: Inspect or not inspect
  - Worker: working costs \( g \), produces value \( v \); gets wage \( w \)
  - Employer: Inspection costs \( h \)
  - We assume \( w > g > h > 0 \)
  - If not inspect → worker shirks → better inspect → if inspect → worker always works → better not inspect → …: No pure NE
  - Employer must randomize
Nash Equilibrium: Non-Existence—of Pure NE—Example 2

Another example: Inspection game

Worker: work or shirk; Employer: Inspect or not inspect

Worker: working costs $g$, produces value $v$; gets wage $w$

Employer: Inspection costs $h$

We assume $w > g > h > 0$

If not inspect $\rightarrow$ worker shirks $\rightarrow$ better inspect $\rightarrow$ if inspect $\rightarrow$ worker always works $\rightarrow$ better not inspect $\rightarrow$ ...: No pure NE

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Games in Strategic Form & Nash Equilibrium

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- Demand for firm 1 is \( D_1(p_1, p_2) = x \) where \( p_1 + tx = p_2 + (1-x) \)
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- \( D_1(p_1, p_2) = 1 - D_2(p_1, p_2) \)
- Nash Equilibrium \((p^\ast_1, p^\ast_2)\): For each \(i: p^\ast_i \in \argmax \{(p_i - c) D_i(p_i, p^\ast_{-i})\}\)
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  + afterwards insert \( r_2(p_1) \) for \( p_2 \)
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  \]
  \[
  p^\ast_1 = p^\ast_2 = c + t \quad \text{for} \quad c + 3/2 \cdot t \leq s
  \]

Indifference condition: more detailed explanation:

For player \(i\)’s utility, we have:

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 u_i(\sigma) = \sum_{s_i \in S_i} \sigma_i(s_i) u_i(s_i, \sigma_{-i}) \quad \text{with} \quad \sum_{s_i \in S_i} \sigma_i(s_i) = 1
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for the NE \(\sigma^*\) we thus have:

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Nash Equilibrium: Non-Existence--of Pure NE--Example 2

- If worker plays \((x, 1-x)\) and employer plays \((y, 1-y)\)
- Indifference condition in mixed strategy NE \(\rightarrow\)
  - \(\rightarrow\) For worker indifferent between S and W:
    - gain from shirking == expected income loss:
      \[
      0y + (1-y)w = y(w-g) + (1-y)(w-g)
      \]
    - \( g = yw \) \(\rightarrow\) \( y = g/w \)
  - \(\rightarrow\) For employer indifferent between L and NL:
    - inspection costs == expected wage savings:
      \[
      x(-h) + (1-x)(v-w-h) = x(-w) + (1-x)(v-w)
      \]
    - \( h = xw \) \(\rightarrow\) \( x = h/w \)
Nash Equilibrium: Non-Existence—of Pure NE: Example 2

- If worker plays \((x, 1-x)\) and employer plays \((y, 1-y)\)
- Indifference condition in mixed strategy NE →
  - \(\rightarrow\) For worker indifferent between \(S\) and \(W\):
gain from shirking = expected income loss:
  \[0y+(1-y)w = y(w-g)+(1-y)(w-g)\]
  \(\rightarrow g = yw \rightarrow y = g/w\)
  - \(\rightarrow\) For employer indifferent between \(I\) and \(NI\):
inspection costs = expected wage savings:
  \[x(-h)+(1-x)(v-w-h) = x(-w) + (1-x)(v-w)\]
  \(\rightarrow h = xw \rightarrow x = h/w\)

Nash Equilibrium: More than one NE

- Another example: Battle of the sexes
  - Man & Woman; Ballet or Football

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- Another example: Game of chicken
  - Driver 1 & Driver 2; Tough or Weak

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Nash Equilibrium: More than one NE

**Another example: Battle of the sexes**
- Two pure NE: (F,F) and (B,B)
- One mixed NE: Indifference condition
  → Let $\sigma_1(F) = x$ and $\sigma_1(B) = y$
  Player 1's indifference:
    $0 \times x + 2(1-x) = 1 \times y + 0(1-y) \Rightarrow y = \frac{2}{3}$
  Player 2's indifference:
    $0 \times x + 2(1-x) = 1 \times 0 + 0(1-x) \Rightarrow x = \frac{2}{3}$
  → Mixed NE: $(\frac{2}{3}, \frac{1}{3}); (\frac{2}{3}, \frac{1}{3})$

**Another example: Game of chicken**
- (same reasoning)
  Mixed NE: $(\frac{1}{2}, 1, 2; \frac{1}{2}, 1, 2)$

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Nash Equilibrium: More than one NE

**Focal points**

- Some games have more than one NE → which will be chosen?
- Theory of "focalness" of NE ("focal points"): Example: Chose time of day simultaneously; reward if match: 12 noon is focal, 15:37 is not

### Risk Dominance

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**Hunt**

Stag Hunt: NE: (C,C) and (D,D); (C,C) is pareto-dominant → (C,C) might be chosen if p(C)>0.5 BUT more than two players: ALL have to agree on C → p(C)<0.5 → p(C)>0.93 → (D,D) "risk dominates" (C,C)
Nash Equilibrium: More than one NE

Focal points

- Some games have more than one NE \(\rightarrow\) which will be chosen?
- Theory of „focalness“ of NE („focal points“): Example: Chose time of day simultaneously; reward if match: 12 noon is focal, 15:37 is not

Risk Dominance

- Stag Hunt: NE: (C;C) and (D;D); (C;C) is pareto-dominant \(\rightarrow\) (C;C) might be chosen if \(p(C)>0.5\)
  BUT
- more than two players: ALL have to agree on C \(\rightarrow p(C)^n>0.5 \rightarrow p(C)>0.93 \rightarrow (D;D)\) „risk dominates“ (C;C)

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