Discussion:

- Integer Linear Programming (ILP) can decide satisfiability of a finite set of equations/inequalities over $\mathbb{Z}$ of the form:
  \[ \sum_{i=1}^{n} a_i \cdot x_i = b \quad \text{bzw.} \quad \sum_{i=1}^{n} a_i \cdot x_i \geq b, \quad a_i \in \mathbb{Z} \]
- Moreover, a (linear) cost function can be optimized :-)
- Warning: The decision problem is in general, already NP-hard !!!
- Notwithstanding that, surprisingly efficient implementations exist.
- Not just loop fusion, but also other re-organizations of loops yield ILP problems ...

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Background 5: Presburger Arithmetic

Many problems in computer science can be formulated without multiplication :-)

Let us first consider two simple special cases ...

1. Linear Equations

\[ \exists x, y, z \cdot \begin{align*}
2x + 3y &= 21 \\
x - y + 5z &= 3
\end{align*} \]

Question:

- Is there a solution over $\mathbb{Q}$ ? \(\n\)
- Is there a solution over $\mathbb{Z}$ ? \(\n\)
- Is there a solution over $\mathbb{N}$ ? \(\n\)

Let us reconsider the equations:

\[ \begin{align*}
2x + 3y &= 24 \\
x - y + 5z &= 3
\end{align*} \]
Answers:

- Is there a solution over $\mathbb{Q}$? Yes
- Is there a solution over $\mathbb{Z}$? No
- Is there a solution over $\mathbb{N}$? No

Complexity:

- Is there a solution over $\mathbb{Q}$? Polynomial
- Is there a solution over $\mathbb{Z}$? Polynomial
- Is there a solution over $\mathbb{N}$? NP-hard

Question:

- Is there a solution over $\mathbb{Q}$? 
- Is there a solution over $\mathbb{Z}$? 
- Is there a solution over $\mathbb{N}$? 

Let us reconsider the equations:

\[
2x + 3y = 24 \\
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\]

Solution Method for Integers:

Observation 1:

\[
a_1x_1 + \ldots + a_kx_k = b \quad (\forall i : a_i \neq 0)
\]

has a solution iff

\[
\gcd(a_1, \ldots, a_k) \mid b
\]

\[
\gcd(a_1, a_2) \mid a_1 x_1 + a_2 x_2 \leq a
\]
Example:

\[ 5y - 10z = 18 \]

has no solution over \( \mathbb{Z} \)  

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Observation 2:

Adding a multiple of one equation to another does not change the set of solutions  

Example:

\[
\begin{align*}
2x + 3y &= 24 \\
x - y + 5z &= 3
\end{align*}
\]

Example:

\[
\begin{align*}
2x + 3y &= 24 \\
x - y + 5z &= 3
\end{align*}
\]

\[
\begin{align*}
5y - 10z &= 18 \\
x - y + 5z &= 3
\end{align*}
\]
Observation 3:

Adding multiples of columns to another column is an invertible transformation which we keep track of in a separate matrix ... 

\[
\begin{array}{c|cc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{array}
\begin{array}{c}
\begin{align}
5y - 10z &= 18 \\
x - y + 5z &= 3 \\
\end{align}
\end{array}
\]

\[
\begin{array}{c|c}
1 & 5y = 18 \\
0 & x - y + 3z = 3 \\
0 & 0 \\
\end{array}
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Example:

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\begin{align}
2x + 3y &= 24 \\
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\]

\[
\begin{array}{c|c}
1 & 5y = 18 \\
0 & x - y = 3 \\
0 & 0 \\
\end{array}
\]

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1 & 5y = 18 \\
0 & x - y = 3 \\
0 & 0 \\
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\]

\[
\begin{array}{c}
\begin{align}
\text{triangular form}!!
\end{align}
\end{array}
\]
Observation 3:

Adding multiples of columns to another column is an invertible transformation which we keep track of in a separate matrix ...
Example

\[
\begin{array}{ccc|c}
1 & 0 & -3 & 5y \\
0 & 1 & 2 & x - y = 3 \\
0 & 0 & 1 &
\end{array}
\]

One special solution:

\[ [6, 3, 0]^T \]

All solutions of the homogeneous system are spanned by:

\[ [0, 0, 1]^T \]

Example

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Solving over \( \mathbb{N} \)

- ... is of major practical importance;
- ... has led to the development of many new techniques;
- ... easily allows to encode NP-hard problems;
- ... remains difficult if just three variables are allowed per equation.
2. **One Polynomial Special Case:**

\[
\begin{align*}
  x & \geq y + 5 \\
  19 & \geq x \\
  y & \geq 13 \\
  y & \geq x - 7
\end{align*}
\]

- There are at most 2 variables per inequation;
- no scaling factors.

---

3. **A General Solution Method:**

**Idea:** Fourier-Motzkin Elimination

- Successively remove individual variables \( x \)!
- All in-equations with **positive** occurrences of \( x \) yield **lower** bounds.
- All in-equations with **negative** occurrences of \( x \) yield **upper** bounds.
- All lower bounds must be at most as big as all upper bounds \( :-) \)
Example:

9 \leq 3x_1 + x_2 \quad (1)  
4 \leq x_1 + 2x_2 \quad (2)  
0 \leq 2x_1 - x_2 \quad (3)  
6 \leq 2x_1 + 6x_2 \quad (4)  
-11 \leq -x_1 - 2x_2 \quad (5)  
-17 \leq -6x_1 + 2x_2 \quad (6)  
-4 \leq -x_2 \quad (7)  

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Idea: Fourier-Motzkin Elimination

- Successively remove individual variables $x$!
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Example:

9 \leq 4x_1 + x_2 \quad (1)  
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-17 \leq -6x_1 + 2x_2 \quad (6)  
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For $x_1$ we obtain:

9 \leq 4x_1 + x_2 \quad (1)  
4 \leq x_1 + 2x_2 \quad (2)  
0 \leq 2x_1 - x_2 \quad (3)  
6 \leq x_1 + 6x_2 \quad (4)  
-11 \leq -x_1 - 2x_2 \quad (5)  
-17 \leq -6x_1 + 2x_2 \quad (6)  
-4 \leq -x_2 \quad (7)  

If such an $x_1$ exists, all lower bounds must be bounded by all upper bounds, i.e.,
This is the one-variable case which we can solve exactly:

Example:

\[
\begin{align*}
9 & \leq 4x_1 + x_2 \quad (1) \\
4 & \leq x_1 + 2x_2 \quad (2) \\
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From which we conclude: \( x_2 \in \left[\frac{7}{2}, 4\right] \) \( \implies \)

In General:

- The original system has a solution over \( \mathbb{Q} \) iff the system after elimination of one variable has a solution over \( \mathbb{Q} \) \( \implies \)
- Every elimination step may square the number of in-equations \( \implies \) exponential run-time \( \implies \)
- It can be modified such that it also decides satisfiability over \( \mathbb{Z} \) \( \implies \) Omega Test
\[
\begin{align*}
\frac{9}{4} - \frac{1}{2}x_2 & \leq 11 - 2x_2 & (1,5) \\
\frac{5}{4} - \frac{1}{2}x_2 & \leq \frac{5}{12} + \frac{1}{3}x_2 & (1,6) \\
4 - 2x_2 & \leq 11 - 2x_2 & (2,5) \\
4 - 2x_2 & \leq \frac{17}{6} + \frac{1}{3}x_2 & (2,6) \\
\frac{1}{2}x_2 & \leq 11 - 2x_2 & (3,5) \\
\frac{1}{2}x_2 & \leq \frac{17}{6} + \frac{1}{3}x_2 & (3,6) \\
6 - 6x_2 & \leq 11 - 2x_2 & (4,5) \\
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-4 & \leq -x_2 & (7)
\end{align*}
\]

**Idea:**

- We successively remove variables. Thereby we omit division ...
- If \( x \) only occurs with coefficient \( \pm 1 \), we apply Fourier-Motzkin elimination \( \therefore \)
- Otherwise, we provide a bound for a positive multiple of \( x \) ...

Consider, e.g., (1) and (6):

\[
\begin{align*}
6 \cdot x_1 & \leq 17 + 2x_2 \\
9 - x_2 & \leq 4 \cdot x_1
\end{align*}
\]

\[
\max \left\{ -1, \frac{1}{2}, -\frac{5}{3}, \frac{1}{3} \right\} \leq x_2 \leq \min \left\{ 5, \frac{22}{3}, 17, \frac{13}{3} \right\}
\]

From which we conclude: \( x_2 \in \left[ \frac{1}{2}, 4 \right] \) \( \therefore \)

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\frac{1}{6}x_2 &\leq 11 - 2x_2 \quad (3.5) \quad \text{or} \quad -22 \leq -5x_2 \quad (3.5) \\
\frac{1}{3}x_2 &\leq \frac{17}{6} + \frac{1}{3}x_2 \quad (3.6) \\
6 - 6x_2 &\leq 11 - 2x_2 \quad (4.5) \\
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W.l.o.g., we only consider strict in-equalities:

\[
\begin{align*}
6 \cdot x_1 &< 18 + 2x_2 \\
8 - x_2 &< 4 \cdot x_1
\end{align*}
\]

... where we always divide by gcds:

\[
\begin{align*}
3 \cdot x_1 &< 9 + x_2 \\
8 - x_2 &< 4 \cdot x_1
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This implies:

\[
3 \cdot (8 - x_2) < 4 \cdot (9 + x_2)
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- We successively remove variables. Thereby we omit division ...
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We thereby obtain:

- If one derived in-equation is **unsatisfiable**, then also the overall system \( \implies \)
- If all derived in-equations are satisfiable, then there is a solution which, however, need not be **integer** \( \implies \)
- An integer solution is guaranteed to exist if there is **sufficient separation** between lower and upper bound \( \implies \)
- Assume \( \alpha < a \cdot x \) \quad \beta \cdot x < \beta \).

Then it should hold that:

\[
\beta \cdot \alpha < a \cdot \beta
\]

and moreover:

\[
[a \cdot b] < a \cdot \beta - b \cdot \alpha
\]
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We thereby obtain:

- If one derived in-equation is unsatisfiable, then also the overall system :-(
- If all derived in-equations are satisfiable, then there is a solution which, however, need not be integer :-(
- An integer solution is guaranteed to exist if there is sufficient separation between lower and upper bound ...
- Assume \( a < a \cdot x \quad b \cdot x < b \).

Then it should hold that:

\[
b \cdot a < a \cdot b
\]

and moreover:

\[
\boxed{a \cdot b} < a \cdot \beta - b \cdot \alpha
\]

... in the Example:

\[
12 < 4 \cdot (9 + x_2) - 3 \cdot (8 - x_2)
\]

or:

\[
12 < 12 + 7x_2
\]

or:

\[
0 < x_2
\]

In the example, also these strengthened in-equations are satisfiable

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\implies \text{the system has a solution over } \mathbb{Z} \implies
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In the example, also these strengthened in-equations are satisfiable

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\]
Discussion:

- If the strengthened in-equations are satisfiable, then also the original system. The reverse implication may be wrong :-(
- In the case where upper and lower bound are not sufficiently separated, we have:
  \[ a \cdot b \leq b \cdot \alpha + a \cdot b \]
  or:
  \[ b \cdot \alpha < ab \cdot x < b \cdot \alpha + a \cdot b \]
Division with \( b \) yields:
  \[ \alpha < a \cdot x < \alpha + a \]
  \[ \alpha + i = a \cdot x \text{ for some } i \in \{1, \ldots, a-1\} \] !!!
Presburger Arithmetic  =  full arithmetic
without multiplication

Arithmetic  :  highly undecidable  :-(
even incomplete  :-(

\[ \exists x_1 \ldots, x_n . \ p(x_1, \ldots, x_n) = 0 \]