... analogously for \( R = M[e]; \) and \( M[e] = e_2; \)

**Transformation 1.2:**

If \( c \) is available at program point \( u \), then \( c \) need not be re-evaluated:

\[ T_c = e; \quad e \in \mathcal{A}[u] \]

We replace the assignment with \( \text{Nop} \) :)
Example:

\[
\begin{align*}
T &= y + 3; \\
x &= y + 3; \\
y &= 7; \\
z &= y + 3;
\end{align*}
\]

Correctness: (Idea)

Transformation 1.1 preserves the semantics and \( A[u] \) for all program points \( u \). :

Assume \( \pi : start \rightarrow^* u \) is the path taken by a computation.
If \( e \in A[u] \), then also \( e \in [\pi]^k \).

Therefore, \( \pi \) can be decomposed into:

\[
\begin{align*}
start \xrightarrow{\pi_1} u_1 & \xrightarrow{k} u_2 \xrightarrow{\pi_2} u_3 \\
& \xrightarrow{\pi_3} \ldots \\
& \xrightarrow{\pi_n} u
\end{align*}
\]

with the following properties:

- The expression \( e \) is evaluated at the edge \( k \);
- The expression \( e \) is not removed from the set of available expressions at any edge in \( \pi_2 \), i.e., no variable of \( e \) receives a new value.

\[
\Rightarrow
\]

The register \( T_e \) contains the value of \( e \) whenever \( u \) is reached. :)}
Warning:

Transformation 1.1 is only meaningful for assignments $x = e$; where:

$\rightarrow x \notin Vars(e)$;
$\rightarrow e \notin Vars$;
$\rightarrow$ the evaluation of $e$ is non-trivial

Which leaves us with the following question ...

Question:

How do we compute $A[u]$ for every program point $u$?

We collect all restrictions to the values of $A[u]$ into a system of constraints:

\[
A[\text{start}] \subseteq \emptyset
\]

\[
A[v] \subseteq [k]^3(A[u])
\]

\[
k = (u, \_, v) \quad \text{edge}
\]

\[
A(x) = \bigcap_{i \in [2]} \{c \in \mathbb{R}^6 \mid c(x) = (u, \_, v)\}
\]
Wanted:
- a maximally large solution
- an algorithm which computes this

Example:

\[ A[0] \subseteq \emptyset \]
\[ A[1] \subseteq (A[0] \cup \{1\}) \setminus Expr_y \]
\[ A[2] \subseteq A[1] \cup \{x > 1\} \]
\[ A[3] \subseteq (A[2] \cup \{x \neq y\}) \setminus Expr_y \]
\[ A[4] \subseteq (A[3] \cup \{y \neq z\}) \setminus Expr_y \]
\[ A[5] \subseteq A[4] \cup \{x > 1\} \]

Solution:

\[ A[0] = \emptyset \]
\[ A[1] = \{1\} \]
\[ A[2] = \{1, x > 1\} \]
\[ A[3] = \{1, x > 1\} \]
\[ A[4] = \{1\} \]
\[ A[5] = \{1, x > 1\} \]
Wanted:

- a maximally large solution
- an algorithm which computes this

Example:

```
0
y = 1;
```

```
1
Neg(x > 1)  Pos(x > 1)
```

```
2
y = x + y;
```

```
3
z = x - 1;
```

```
4
```

Solution:

- \( A[0] = \emptyset \)
- \( A[1] = \{1\} \)
- \( A[2] = \{1, x > 1\} \)
- \( A[3] = \{1, x > 1\} \)
- \( A[4] = \{1\} \)
- \( A[5] = \{1, x > 1\} \)

Observation:

- The possible values for \( A[a] \) form a complete lattice:
  \( D = 2^{\text{Expr}} \) with \( B_1 \sqsubseteq B_2 \) iff \( B_1 \supseteq B_2 \)

Observation:

- The functions \( k^1 : D \rightarrow D \) are monotonic, i.e.,
  \[ [k^1]^2(B_1) \subseteq [k^2]^2(B_2) \]
  whenever \( B_1 \sqsubseteq B_2 \)
**Background 2: Complete Lattices**

A set $D$ together with a relation $\subseteq D \times D$ is a partial order if for all $a, b, c \in D$:

- Reflexivity: $a \preceq a$
- Anti-symmetry: $a \preceq b \land b \preceq a \implies a = b$
- Transitivity: $a \preceq b \land b \preceq c \implies a \preceq c$

**Examples:**

1. $D = 2^{\{a, b, c\}}$ with the relation $\subseteq$:

2. $\mathbb{Z}$ with the relation $=:$

3. $\mathbb{Z}$ with the relation $\preceq$:

4. $\mathbb{Z}_+ = \mathbb{Z} \cup \{\perp\}$ with the ordering:

$d \in D$ is called upper bound for $X \subseteq D$ if

$$x \preceq d \quad \text{for all } x \in X$$

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$d$ is called least upper bound (lub) if

1. $d$ is an upper bound and
2. $d \preceq y$ for every upper bound $y$ of $X$. 
A complete lattice (cl) \( \mathcal{D} \) is a partial ordering where every subset \( X \subseteq \mathcal{D} \) has a least upper bound \( \bigvee X \in \mathcal{D} \).

Note:

Every complete lattice has
- a least element \( \bot = \bigvee \emptyset \in \mathcal{D} \);
- a greatest element \( \top = \bigvee \mathcal{D} \in \mathcal{D} \).

Examples:

1. \( \mathcal{D} = 2^{(a,b,c)} \) is a cl :)
2. \( \mathcal{D} = \mathbb{Z} \) with “=” is not.
3. \( \mathcal{D} = \mathbb{Z} \) with “≤” is neither.
4. \( \mathcal{D} = \mathbb{Z}_\bot \) is also not :-(
5. With an extra element \( T \), we obtain the flat lattice \( \mathbb{Z}_T^\top = \mathbb{Z} \cup \{ \bot, T \} \):

\[
\begin{array}{c}
\cdots 2 \ 3 \ 0 \ 1 \ 2 \ \\
\bigvee \mathcal{D} \in \mathcal{D}.
\end{array}
\]
We have:

**Theorem:**

If \( D \) is a complete lattice, then every subset \( X \subseteq D \) has a greatest lower bound \( \bigcap X \).

Proof:

Construct \( U = \{ u \in D | \forall x \in X : u \subseteq x \} \).

\[ \bigcap\text{ the set of all lower bounds of } X \]
We have:

**Theorem:**

If $\mathbb{D}$ is a complete lattice, then every subset $X \subseteq \mathbb{D}$ has a greatest lower bound $\bigwedge X$.

**Proof:**

Construct $U = \{ u \in \mathbb{D} \mid \forall x \in X : u \sqsubseteq x \}$.  
// the set of all lower bounds of $X$  :-)

(1) $g$ is a lower bound of $X$:

Assume $x \in X$. Then:

$\forall u \in U$ $u \sqsubseteq x$  
$\implies$ $x$ is an upper bound of $U$  
$\implies$ $g \sqsubseteq x$  :-)

We have:

**Theorem:**

If $\mathbb{D}$ is a complete lattice, then every subset $X \subseteq \mathbb{D}$ has a greatest lower bound $\bigwedge X$.

**Proof:**

Construct $U = \{ u \in \mathbb{D} \mid \forall x \in X : u \sqsubseteq x \}$.  
// the set of all lower bounds of $X$  :-)

Set: $g := \bigcup U$

Claim: $g = \bigwedge X$
(1) \textit{\textbf{g} is a lower bound of \(X\):}

Assume \(x \in X\). Then:
\begin{itemize}
\item \(u \subseteq x\) for all \(u \in U\)
\item \(x\) is an upper bound of \(U\)
\item \(\therefore g \subseteq x\) :)
\end{itemize}

(2) \textit{\textbf{g} is the greatest lower bound of \(X\):}

Assume \(u\) is a lower bound of \(X\). Then:
\begin{itemize}
\item \(u \in U\)
\item \(\therefore u \subseteq g\) :))
\end{itemize}

We are looking for solutions for systems of constraints of the form:
\[ x_i \equiv f_i(x_1, \ldots, x_n) \quad (\ast) \]

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\[ x_i \equiv f_i(x_1, \ldots, x_n) \quad (\ast) \]

where:
\begin{tabular}{|c|c|c|c|c|}
\hline
\(x_i\) & unknown & here & \(A[i]\) \\
\hline
\(D\) & values & here & \(A\) & \(\mathbb{E}\) \\
\hline
\(\subseteq\) & ordering relation & here & \(\preceq\) \\
\hline
\(f_i: D^n \rightarrow D\) & constraint & here & \ldots \\
\hline
\end{tabular}
We are looking for solutions for systems of constraints of the form:

\[ x_i \supseteq f_i(x_1, \ldots, x_n) \quad (\ast) \]

where:

<table>
<thead>
<tr>
<th>( x_i )</th>
<th>unknown</th>
<th>here: ( A[u] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D )</td>
<td>values</td>
<td>here: ( 2^{E_{\text{expr}}} )</td>
</tr>
<tr>
<td>( \subseteq \subseteq D \times D )</td>
<td>ordering relation</td>
<td>here: ( \supseteq )</td>
</tr>
<tr>
<td>( f_i: D^n \rightarrow D )</td>
<td>constraint</td>
<td>here: ...</td>
</tr>
</tbody>
</table>

Constraint for \( A[v] \) \((v \neq \text{start})\):

\[ A[v] \subseteq \bigcap \{ [k]_F(A[u]) \mid k = (u, v) \text{ edge} \} \]

A mapping \( f : D_1 \rightarrow D_2 \) is called monotonic if \( f(a) \subseteq f(b) \) for all \( a \subseteq b \).

Because:

\[ x \supseteq d_1 \wedge \ldots \wedge x \supseteq d_k \quad \text{iff} \quad x \supseteq \bigcup \{d_1, \ldots, d_k\} \quad :\)