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Title: Seidl: Programmoptimierung (09.01.2012)

Date: Mon Jan 09 12:33:43 CET 2012

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Pages: 72

Discussion:

- Solutions only matter within the bounds to the iteration variables.
- Every integer solution there provides a conflict.
- Fusion of loops is possible if no conflicts occur :-)
- The given secial cases suffice to solve the case of two variables over $\mathbb{Q}$ and of one variable over $\mathbb{Z}$:-)
- The number of variables in the inequations corresponds to the nesting-depth of for-loops $\Longrightarrow$ in general, is quite small :-)

Discussion:

- Integer Linear Programming (ILP) can decide satisfiability of a finite set of equations/inequations over $\mathbb{Z}$ of the form:

$$\sum_{i=1}^{n} a_i \cdot x_i = b \quad \text{bzw.} \quad \sum_{i=1}^{n} a_i \cdot x_i \geq b\,, \quad a_i \in \mathbb{Z}$$

- Moreover, a (linear) cost function can be optimized :-)
- Warning: The decision problem is in general, already NP-hard !!!
- Notwithstanding that, surprisingly efficient implementations exist.
- Not just loop fusion, but also other re-organizations of loops yield ILP problems ...

Background 5: Presburger Arithmetic

Many problems in computer science can be formulated without multiplication :-)

Let us first consider two simple special cases ...

1. Linear Equations

$$\begin{array}{rcrcrcr} 2x & + & 3y & & & = & 24 \\ x & - & y & + & 5z & = & 3 \end{array}$$

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Question:

- Is there a solution over $\mathbb{Q}$?
- Is there a solution over $\mathbb{Z}$?
- Is there a solution over $\mathbb{N}$?

?

Let us reconsider the equations:

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Answers:

- Is there a solution over $\mathbb{Q}$? Yes
- Is there a solution over $\mathbb{Z}$? No
- Is there a solution over $\mathbb{N}$? No

Complexity:

- Is there a solution over $\mathbb{Q}$? Polynomial
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$$5y - 10z = 18$$

has no solution over $\mathbb{Z}$:-)

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Observation 2:

Adding a multiple of one equation to another does not change the set of solutions :-)

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Adding multiples of columns to another column is an invertible transformation which we keep track of in a separate matrix ...

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$\Longrightarrow$ triangular form !!

Observation 4:

- A special solution of a triangular system can be directly read off :-)
- All solutions of a homogeneous triangular system can be directly read off :-)
- All solutions of the original system can be recovered from the solutions of the triangular system by means of the accumulated transformation matrix:-))

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Example

$$\begin{array}{rrr|rcrcr} 1 & 0 & -3 & & & 5y & = & 15 \\ 0 & 1 & 2 & x & - & y & = & 3 \\ 0 & 0 & 1 & & & & & \end{array}$$

One special solution:

$$[6, 3, 0]^\top$$

All solutions of the homogeneous system are spanned by:

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Solving over $\mathbb{N}$

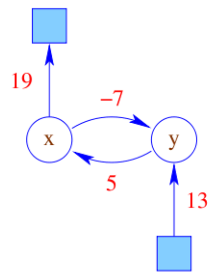
- ... is of major practical importance;
- ... has led to the development of many new techniques;
- ... easily allows to encode NP-hard problems;
- ... remains difficult if just three variables are allowed per equation.

2. One Polynomial Special Case:

$$
\begin{aligned}
x &\geq y + 5 \\
19 &\geq x \\
y &\geq 13 \\
y &\geq x - 7
\end{aligned}
$$

- There are at most 2 variables per in-equation;
- no scaling factors.

Idea: Represent the system by a graph:



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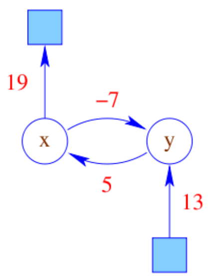

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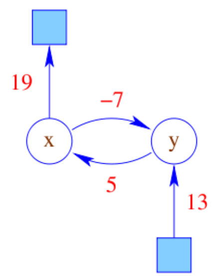

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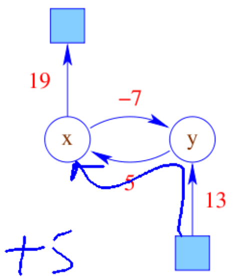

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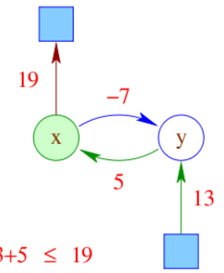

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The in-equations are satisfiable iff

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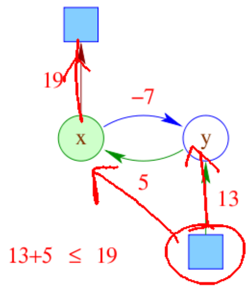
Compute the reflexive and transitive closure of the edge weights!



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→ Bellman-Ford

3. A General Solution Method:

Idea: Fourier-Motzkin Elimination

- Successively remove individual variables x !
- All in-equations with positive occurrences of x yield lower bounds.
- All in-equations with negative occurrences of x yield upper bounds.
- All lower bounds must be at most as big as all upper bounds ;-))

Jean Baptiste Joseph Fourier, 1768–1830

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Example:

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\begin{array}{rcll}
9 & \leq & 4x_1 + x_2 & (1) \\
4 & \leq & x_1 + 2x_2 & (2) \\
0 & \leq & 2x_1 - x_2 & (3) \\
6 & \leq & x_1 + 6x_2 & (4) \\
-11 & \leq & -x_1 - 2x_2 & (5) \\
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\end{array}
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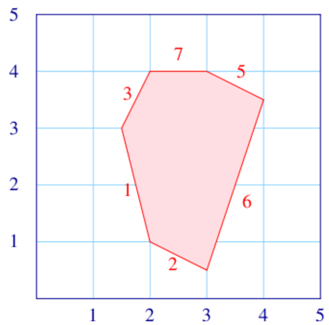

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For x_1 we obtain:

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\qquad\qquad
\begin{array}{rcll}
\frac{9}{4} - \frac{1}{4}x_2 & \leq & x_1 & (1) \\
4 - 2x_2 & \leq & x_1 & (2) \\
\frac{1}{2}x_2 & \leq & x_1 & (3) \\
6 - 6x_2 & \leq & x_1 & (4) \\
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If such an x_1 exists, all lower bounds must be bounded by all upper bounds, i.e.,

$$
\begin{array}{rcll}
\frac{9}{4} - \frac{1}{4}x_2 & \leq & 11 - 2x_2 & (1,5) \\
\frac{9}{4} - \frac{1}{4}x_2 & \leq & \frac{17}{6} + \frac{1}{3}x_2 & (1,6) \\
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-4 & \leq & -x_2 & (7)
\end{array}
\quad \text{or} \quad
\begin{array}{rcll}
-35 & \leq & -7x_2 & (1,5) \\
-\frac{7}{12} & \leq & \frac{7}{12}x_2 & (1,6) \\
-7 & \leq & 0 & (2,5) \\
\frac{7}{6} & \leq & \frac{7}{3}x_2 & (2,6) \\
-22 & \leq & -5x_2 & (3,5) \\
-\frac{17}{6} & \leq & -\frac{1}{6}x_2 & (3,6) \\
-5 & \leq & 4x_2 & (4,5) \\
\frac{19}{6} & \leq & \frac{19}{3}x_2 & (4,6) \\
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$$

$$
\max\ \{-1, \boxed{\tfrac{1}{2}}, -\tfrac{5}{4}, \tfrac{1}{2}\} \quad \leq \quad x_2 \quad \leq \quad \min\ \{5, \tfrac{22}{5}, 17, \boxed{4}\}
$$

From which we conclude: $x_2 \in [\frac{1}{2}, 4]$:-)

In General:

- The original system has a solution over $\mathbb{Q}$ iff the system after elimination of one variable has a solution over $\mathbb{Q}$:-)
- Every elimination step may square the number of in-equations
  $\Longrightarrow$ exponential run-time :-((
- It can be modified such that it also decides satisfiability over $\mathbb{Z}$
  $\Longrightarrow$ Omega Test

William Worthington Pugh, Jr.
University of Maryland, College Park

Idea:

- We successively remove variables. Thereby we omit division ...
- If x only occurs with coefficient ± 1, we apply Fourier-Motzkin elimination :-)
- Otherwise, we provide a bound for a positive multiple of x ...

Consider, e.g., (1) and (6) :

$$\begin{array}{rcl} 6 \cdot x_1 & \leq & 17 + 2x_2 \\ 9 - x_2 & \leq & 4 \cdot x_1 \end{array}$$

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W.l.o.g., we only consider strict in-equations:

$$\begin{array}{rcl} 6 \cdot x_1 & < & 18 + 2x_2 \\ 8 - x_2 & < & 4 \cdot x_1 \end{array}$$

... where we always divide by gcds:

$$\begin{array}{rcl} 3 \cdot x_1 & < & 9 + x_2 \\ 8 - x_2 & < & 4 \cdot x_1 \end{array}$$

This implies:

$$3 \cdot (8 - x_2) < 4 \cdot (9 + x_2)$$

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We thereby obtain:

- If one derived in-equation is unsatisfiable, then also the overall system :-)
- If all derived in-equations are satisfiable, then there is a solution which, however, need not be integer :-(
- An integer solution is guaranteed to exist if there is sufficient separation between lower and upper bound ...
- Assume $\alpha < a \cdot x \qquad b \cdot x < \beta$.

  Then it should hold that:

  $$b \cdot \alpha < a \cdot \beta$$

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  $$\boxed{a \cdot b} < a \cdot \beta - b \cdot \alpha$$

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... in the Example:

$$12 \quad < \quad 4 \cdot (9 + x_2) - 3 \cdot (8 - x_2)$$

or:

$$12 \quad < \quad 12 + 7x_2$$

or:

$$0 \quad < \quad x_2$$

In the example, also these strengthened in-equations are satisfiable

$\Longrightarrow$ the system has a solution over $\mathbb{Z}$:-)

Discussion:

- If the strengthened in-equations are satisfiable, then also the original system. The reverse implication may be wrong :-(
- In the case where upper and lower bound are not sufficiently separated, we have:

  $$a \cdot \beta \leq b \cdot \alpha + \boxed{a \cdot b}$$

  or:

  $$\cancel{b} \cdot \alpha < \boxed{a\cancel{b} \cdot x} < \cancel{b} \cdot \alpha + \boxed{a \cdot \cancel{b}}$$

  Division with b yields:

  $$\alpha < a \cdot x < \alpha + \boxed{a}$$

  $\Longrightarrow$ $\boxed{\alpha + i = a \cdot x}$ for some $i \in \{1, \ldots, a - 1\}$!!!

Discussion (cont.):

→ Fourier-Motzkin Elimination is not the best method for rational systems of in-equations.

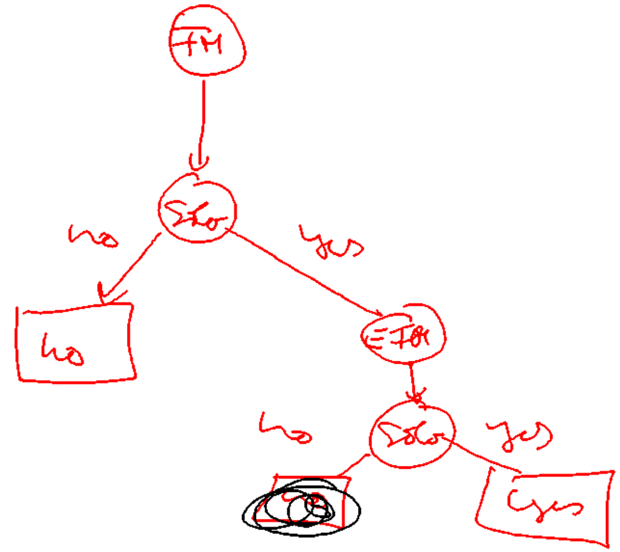
→ The Omega test is necessarily exponential :-)

If the system is solvable, the test generally terminates rapidly.

It may have problems with unsolvable systems :-(

→ Also for ILP, there are other/smarter algorithms ...

→ For programming language problems, however, it seems to behave quite well :-)



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4. Generalization to a Logic

Disjunction:

$$\begin{array}{llll} (x - 2y = 15 & \wedge & x + y = 7) & \vee \\ (x + y = 6 & \wedge & 3x + z = -8) & \end{array}$$

Quantors:

$$\exists\, x : \quad z - 2x = 42 \quad \wedge \quad z + x = 19$$

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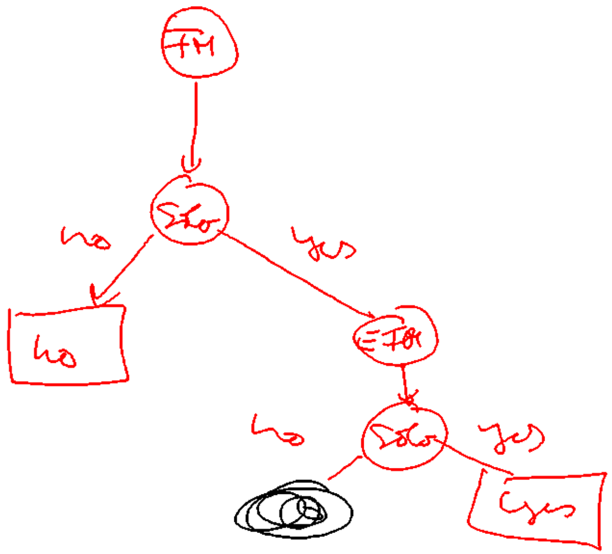
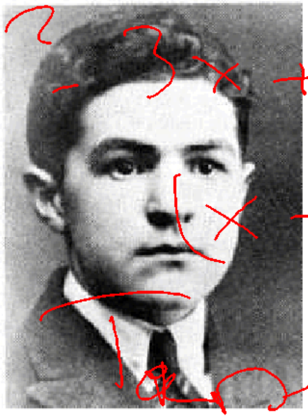
$\Longrightarrow$ Presburger Arithmetic

Mojzesz Presburger, 1904–1943 (?)



$\exists x.\ x^2 - 3x + 5 \geq (x-5)^2$



$\log_2$

Mojzesz Presburger, 1904–1943 (?)

$\mathbb{R}$

Presburger Arithmetic = full arithmetic without multiplication

Arithmetic : highly undecidable :-(

even incomplete :-((

$\Longrightarrow$ Hilbert's 10th Problem

$\Longrightarrow$ Gödel's Theorem

Presburger Formulas over $\mathbb{N}$:

$$\phi \quad ::= \quad x + y = z \quad | \quad x = n \quad | \quad \phi_1 \wedge \phi_2 \quad | \quad \neg\phi \quad | \quad \exists\, x : \ \phi$$

$\forall x.\varphi \equiv \neg(\exists x.\neg\varphi)$

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Find values for the free variables in $\mathbb{N}$ such that ϕ holds ...

Idea: Code the values of the variables as Words :-)

213	t	1	0	1	0	1	0	1	1
42	z	0	1	0	1	0	1	0	0
89	y	1	0	0	1	1	0	1	0
17	x	1	0	0	0	1	0	0	0

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213	t	1	0	1	0	1	0	1	1
42	z	0	1	0	1	0	1	0	0
89	y	1	0	0	1	1	0	1	0
17	x	1	0	0	0	1	0	0	0

Idea: Code the values of the variables as Words :-)

213	t	1	0	1	0	1	0	1	1
42	z	0	1	0	1	0	1	0	0
89	y	1	0	0	1	1	0	1	0
17	x	1	0	0	0	1	0	0	0

Observation:

The set of satisfying variable assignments is regular :-))

$\phi_1 \wedge \phi_2$	$\Longrightarrow$	$\mathcal{L}(\phi_1) \cap \mathcal{L}(\phi_2)$	(Intersection)
$\neg\phi$	$\Longrightarrow$	$\overline{\mathcal{L}(\phi)}$	(Complement)
$\exists\, x : \quad \phi$	$\Longrightarrow$	$\pi_x(\mathcal{L}(\phi))$	(Projection)

Projecting away the x-component:

213	t	1	0	1	0	1	0	1	1
42	z	0	1	0	1	0	1	0	0
89	y	1	0	0	1	1	0	1	0
17	x	1	0	0	0	1	0	0	0

Approximation of Horn Clauses

Step 1:

Simplification of pre-conditions by splitting, simplification and guard simplification (as before :-)

Step 2:

Introduction of copies of variables X. Every copy receives all literals of X as pre-condition.

$$p(f(X,X)) \quad \leftarrow \quad q(X) \qquad \text{yields :}$$

$$p(f(X,X')) \quad \leftarrow \quad q(X), q(X')$$

Assume $A = \{p_1, \ldots, p_m\}$. Then:

$$[A](b) \quad \leftarrow \qquad \text{whenever} \quad p_i(b) \leftarrow \quad \text{for all } i.$$

$$[A](f(X_1,\ldots,X_k)) \quad \leftarrow \quad [B_1](X_1),\ldots,[B_k](X_k)$$

$$\text{whenever} \quad B_i = \{p_{jl} \mid X_{i_{jl}} = X_i\} \quad \text{for}$$

$$p_j(f(X_1,\ldots,X_k)) \leftarrow p_{j1}(X_{i_{j1}}),\ldots,p_{jr_j}(X_{i_{jr_j}})$$

Warning:

Automata for Basic Predicates:

$x = 5$