

Script generated by TTT

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81

2-3 Trees

```
datatype 'a tree23 = ⟨  
  | Node2 ('a tree23) 'a ('a tree23)  
  | Node3 ('a tree23) 'a ('a tree23) 'a ('a tree23)
```

83

isin

```
isin ⟨l, a, m, b, r⟩ x =  
(case cmp x a of  
  | LT ⇒ isin l x  
  | EQ ⇒ True  
  | GT ⇒ case cmp x b of  
    | LT ⇒ isin m x  
    | EQ ⇒ True  
    | GT ⇒ isin r x)
```

84

Structural invariant bal

All leaves are at the same level:

85

Structural invariant bal

All leaves are at the same level:

$$bal \langle \rangle = True$$

$$bal \langle l, -, r \rangle = (bal \ l \wedge bal \ r \wedge h(l) = h(r))$$

$$bal \langle l, -, m, -, r \rangle = \\ (bal \ l \wedge bal \ m \wedge bal \ r \wedge h(l) = h(m) \wedge h(m) = h(r))$$

85

Structural invariant bal

All leaves are at the same level:

$$bal \langle \rangle = True$$

$$bal \langle l, -, r \rangle = (bal \ l \wedge bal \ r \wedge h(l) = h(r))$$

$$bal \langle l, -, m, -, r \rangle = \\ (bal \ l \wedge bal \ m \wedge bal \ r \wedge h(l) = h(m) \wedge h(m) = h(r))$$

Lemma

$$bal \ t \implies 2^{h(t)} \leq |t| + 1$$

85

Insertion

The idea:

$Leaf \rightsquigarrow Node2$

$Node2 \rightsquigarrow Node3$

$Node3 \rightsquigarrow \text{overflow, pass 1 element back up}$

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Insertion

Two possible return values:

- tree accommodates new element without increasing height: $T_i t$

87

Insertion

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- tree accommodates new element without increasing height: $T_i t$
- tree overflows: $Up_i l x r$

87

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datatype 'a up_i = T_i ('a tree23)
| Up_i ('a tree23) 'a ('a tree23)

87

Insertion

Two possible return values:

- tree accommodates new element without increasing height: $T_i t$
- tree overflows: $Up_i l x r$

datatype 'a up_i = T_i ('a tree23)
| Up_i ('a tree23) 'a ('a tree23)

$tree_i :: 'a up_i \Rightarrow 'a tree23$

87

Insertion

Two possible return values:

- tree accommodates new element without increasing height: $T_i t$
- tree overflows: $Up_i l x r$

datatype $'a\ up_i = T_i ('a\ tree23)$
| $Up_i ('a\ tree23) 'a ('a\ tree23)$

$tree_i :: 'a\ up_i \Rightarrow 'a\ tree23$

$tree_i (T_i t) = t$

$tree_i (Up_i l a r) = \langle l, a, r \rangle$

87

Insertion

$insert :: 'a \Rightarrow 'a\ tree23 \Rightarrow 'a\ tree23$

$insert\ x\ t = tree_i (ins\ x\ t)$

88

Insertion

Two possible return values:

- tree accommodates new element without increasing height: $T_i t$
- tree overflows: $Up_i l x r$

datatype $'a\ up_i = T_i ('a\ tree23)$
| $Up_i ('a\ tree23) 'a ('a\ tree23)$

$tree_i :: 'a\ up_i \Rightarrow 'a\ tree23$

$tree_i (T_i t) = t$

$tree_i (Up_i l a r) = \langle l, a, r \rangle$

87

Insertion

$insert :: 'a \Rightarrow 'a\ tree23 \Rightarrow 'a\ tree23$

$insert\ x\ t = tree_i (ins\ x\ t)$

$ins :: 'a \Rightarrow 'a\ tree23 \Rightarrow 'a\ up_i$

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Insertion

$$\text{ins } x \langle \rangle = \text{Up}_i \langle \rangle x \langle \rangle$$

89

Insertion

$$\begin{aligned} \text{ins } x \langle \rangle &= \text{Up}_i \langle \rangle x \langle \rangle \\ \text{ins } x \langle l, a, r \rangle &= \end{aligned}$$

89

Insertion

$$\begin{aligned} \text{ins } x \langle \rangle &= \text{Up}_i \langle \rangle x \langle \rangle \\ \text{ins } x \langle l, a, r \rangle &= \\ \text{case } \text{cmp } x \ a \ \text{of} & \\ \quad \text{LT} \Rightarrow \text{case } \text{ins } x \ l \ \text{of} & \\ \quad \quad T_i \ l' \Rightarrow T_i \langle l', a, r \rangle & \\ \quad \quad | \ \text{Up}_i \ l_1 \ b \ l_2 \Rightarrow T_i \langle l_1, b, l_2, a, r \rangle & \\ | \ \text{EQ} \Rightarrow T_i \langle l, x, r \rangle & \\ | \ \text{GT} \Rightarrow \text{case } \text{ins } x \ r \ \text{of} & \\ \quad \quad T_i \ r' \Rightarrow T_i \langle l, a, r' \rangle & \\ \quad \quad | \ \text{Up}_i \ r_1 \ b \ r_2 \Rightarrow T_i \langle l, a, r_1, b, r_2 \rangle & \end{aligned}$$

89

Insertion

$$\text{ins } x \langle l, a, m, b, r \rangle =$$

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Insertion

```
ins x ⟨l, a, m, b, r⟩ =
case cmp x a of
  LT ⇒ case ins x l of
    Ti l' ⇒ Ti ⟨l', a, m, b, r⟩
    | Upi l1 c l2 ⇒ Upi ⟨l1, c, l2⟩ a ⟨m, b, r⟩
  | EQ ⇒ Ti ⟨l, a, m, b, r⟩
  | GT ⇒
    case cmp x b of
      LT ⇒
        case ins x m of
          Ti m' ⇒ Ti ⟨l, a, m', b, r⟩
          | Upi m1 c m2 ⇒ Upi ⟨l, a, m1⟩ c ⟨m2, b, r⟩
        | EQ ⇒ Ti ⟨l, a, m, b, r⟩
        | GT ⇒
```

90

Insertion preserves *bal*

Lemma

$bal\ t \implies bal\ (tree_i\ (ins\ a\ t))$

91

Insertion

```
ins x ⟨l, a, m, b, r⟩ =
```

90

Insertion preserves *bal*

Lemma

$bal\ t \implies bal\ (tree_i\ (ins\ a\ t))$

Proof by induction on t .

91

Insertion

```
ins x ⟨⟩ = Upi ⟨⟩ x ⟨⟩
ins x ⟨l, a, r⟩ =
case cmp x a of
  LT ⇒ case ins x l of
    Ti l' ⇒ Ti ⟨l', a, r⟩
    | Upi l1 b l2 ⇒ Ti ⟨l1, b, l2, a, r⟩
  | EQ ⇒ Ti ⟨l, x, r⟩
  | GT ⇒ case ins x r of
    Ti r' ⇒ Ti ⟨l, a, r'⟩
    | Upi r1 b r2 ⇒ Ti ⟨l, a, r1, b, r2⟩
```

89

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$bal\ t \implies bal\ (tree_i\ (ins\ a\ t))$

91

Insertion

```
ins x ⟨⟩ = Upi ⟨⟩ x ⟨⟩
ins x ⟨l, a, r⟩ =
case cmp x a of
  LT ⇒ case ins x l of
    Ti l' ⇒ Ti ⟨l', a, r⟩
    | Upi l1 b l2 ⇒ Ti ⟨l1, b, l2, a, r⟩
  | EQ ⇒ Ti ⟨l, x, r⟩
  | GT ⇒ case ins x r of
    Ti r' ⇒ Ti ⟨l, a, r'⟩
    | Upi r1 b r2 ⇒ Ti ⟨l, a, r1, b, r2⟩
```

89

Insertion preserves *bal*

Lemma

$bal\ t \implies bal\ (tree_i\ (ins\ a\ t))$

91

Insertion preserves *bal*

Lemma

$bal\ t \implies bal\ (tree_i\ (ins\ a\ t))$

Proof by induction on t .

91

Insertion preserves *bal*

Lemma

$bal\ t \implies bal\ (tree_i\ (ins\ a\ t))$

where $h :: 'a\ up_i \Rightarrow nat$

$h(T_i\ t) = h(t)$

$h(Up_i\ l\ a\ r) = h(l)$

Proof by induction on t .

91

Insertion preserves *bal*

Lemma

$bal\ t \implies bal\ (tree_i\ (ins\ a\ t)) \wedge h(ins\ a\ t) = h(t)$

where $h :: 'a\ up_i \Rightarrow nat$

$h(T_i\ t) = h(t)$

$h(Up_i\ l\ a\ r) = h(l)$

Proof by induction on t .

91

Insertion preserves *bal*

Lemma

$bal\ t \implies bal\ (tree_i\ (ins\ a\ t)) \wedge h(ins\ a\ t) = h(t)$

where $h :: 'a\ up_i \Rightarrow nat$

$h(T_i\ t) = h(t)$

$h(Up_i\ l\ a\ r) = h(l)$

Proof by induction on t . Base and step automatic.

Corollary

$bal\ t \implies bal\ (insert\ a\ t)$

91

Deletion

The idea:

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Deletion

Two possible return values:

- height unchanged: $T_d t$
- height decreased by 1: $Up_d t$

datatype $'a\ up_d = T_d ('a\ tree23) \mid Up_d ('a\ tree23)$

$tree_d (T_d t) = t$

$tree_d (Up_d t) = t$

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Deletion

$delete :: 'a \Rightarrow 'a\ tree23 \Rightarrow 'a\ tree23$

94

Deletion

$delete :: 'a \Rightarrow 'a\ tree23 \Rightarrow 'a\ tree23$
 $delete\ x\ t = tree_d (del\ x\ t)$

94

Deletion

```
delete :: 'a ⇒ 'a tree23 ⇒ 'a tree23
delete x t = tree_d (del x t)

del :: 'a ⇒ 'a tree23 ⇒ 'a up_d
```

94

```
del x ⟨l, a, r⟩ =
(case cmp x a of
  LT ⇒ node21 (del x l) a r
  | EQ ⇒ let (a', t) = split_min r in node22 l a' t
  | GT ⇒ node22 l a (del x r))
```

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```
del x ⟨l, a, r⟩ =
(case cmp x a of
  LT ⇒ node21 (del x l) a r
  | EQ ⇒ let (a', t) = split_min r in node22 l a' t
  | GT ⇒ node22 l a (del x r))
```

96

```
del x ⟨l, a, r⟩ =
(case cmp x a of
  LT ⇒ node21 (del x l) a r
  | EQ ⇒ let (a', t) = split_min r in node22 l a' t
  | GT ⇒ node22 l a (del x r))
```

```
node21 (T_d t1) a t2 = T_d ⟨t1, a, t2⟩
node21 (Up_d t1) a ⟨t2, b, t3⟩ = Up_d ⟨t1, a, t2, b, t3⟩
node21 (Up_d t1) a ⟨t2, b, t3, c, t4⟩ =
T_d ⟨⟨t1, a, t2⟩, b, ⟨t3, c, t4⟩⟩
```

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Deletion preserves *bal*

After 13 simple lemmas:

Lemma

$bal\ t \implies bal\ (tree_d\ (del\ x\ t))$

Corollary

$bal\ t \implies bal\ (delete\ x\ t)$

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Beyond 2-3 trees

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Beyond 2-3 trees

datatype 'a tree234 =

Leaf | *Node2* ... | *Node3* ... | *Node4* ...

Like 2-3 trees, but with many more cases

The general case:

B-trees and (a, b) -trees

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- ⑧ Unbalanced BST
- ⑨ Abstract Data Types
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- ⑪ Red-Black Trees
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99

Relationship to 2-3-4 trees

Idea: encode 2-3-4 trees as binary trees;

101

Relationship to 2-3-4 trees

Idea: encode 2-3-4 trees as binary trees;
use color to express grouping

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use color to express grouping

$$\langle \rangle \approx \langle \rangle$$

101

Relationship to 2-3-4 trees

Idea: encode 2-3-4 trees as binary trees;
use color to express grouping

$$\begin{aligned} \langle \rangle &\approx \langle \rangle \\ \langle t_1, a, t_2 \rangle &\approx \langle t_1, a, t_2 \rangle \\ \langle t_1, a, t_2, b, t_3 \rangle &\approx \langle \langle t_1, a, t_2 \rangle, b, t_3 \rangle \text{ or } \langle t_1, a, \langle t_2, b, t_3 \rangle \rangle \end{aligned}$$

101

Relationship to 2-3-4 trees

Idea: encode 2-3-4 trees as binary trees;
use color to express grouping

$$\begin{aligned} \langle \rangle &\approx \langle \rangle \\ \langle t_1, a, t_2 \rangle &\approx \langle t_1, a, t_2 \rangle \\ \langle t_1, a, t_2, b, t_3 \rangle &\approx \langle \langle t_1, a, t_2 \rangle, b, t_3 \rangle \text{ or } \langle t_1, a, \langle t_2, b, t_3 \rangle \rangle \\ \langle t_1, a, t_2, b, t_3, c, t_4 \rangle &\approx \langle \langle t_1, a, t_2 \rangle, b, \langle t_3, c, t_4 \rangle \rangle \end{aligned}$$

Red means "I am part of a bigger node"

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Structural invariants

- The root is

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Structural invariants

- The root is Black.
- Every $\langle \rangle$ is considered Black.

102

Structural invariants

- The root is Black.
- Every $\langle \rangle$ is considered Black.
- If a node is Red,

102

Structural invariants

- The root is Black.
- Every $\langle \rangle$ is considered Black.
- If a node is Red, its children are Black.
- All paths from a node to a leaf have the same number of

102

Red-black trees

datatype *color* = *Red* | *Black*

datatype

'a rbt = *Leaf* | *Node color ('a tree) 'a ('a tree)*

103

Red-black trees

datatype *color* = *Red* | *Black*

datatype

'a rbt = *Leaf* | *Node color ('a tree) 'a ('a tree)*

Abbreviations:

$\langle \rangle \equiv \text{Leaf}$
 $\langle l, a, c, r \rangle \equiv \text{Node } l \ a \ c \ r$

103

Red-black trees

datatype *color* = *Red* | *Black*

datatype

'a rbt = *Leaf* | *Node color ('a tree) 'a ('a tree)*

Abbreviations:

$\langle \rangle \equiv \text{Leaf}$
 $\langle l, a, c, r \rangle \equiv \text{Node } l \ a \ c \ r$
 $R \ l \ a \ r \equiv \text{Node } l \ a \ \text{Red } r$

103

Red-black trees

datatype *color* = *Red* | *Black*

datatype

'a rbt = *Leaf* | *Node color ('a tree) 'a ('a tree)*

Abbreviations:

$\langle \rangle \equiv \text{Leaf}$
 $\langle l, a, c, r \rangle \equiv \text{Node } l \ a \ c \ r$
R l a r \equiv *Node l a Red r*
B l a r \equiv *Node l a Black r*

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Structural invariants

rbt :: *'a rbt* \Rightarrow *bool*

rbt t = (*invc t* \wedge *invh t* \wedge *color t* = *Black*)

105

Structural invariants

105

Red-black trees

datatype *color* = *Red* | *Black*

datatype

'a rbt = *Leaf* | *Node color ('a tree) 'a ('a tree)*

103

Structural invariants

$invh :: 'a\ rbt \Rightarrow bool$

106

Structural invariants

$invh :: 'a\ rbt \Rightarrow bool$

$invh\ \langle \rangle = True$

$invh\ \langle l, _ , _ , r \rangle = (invh\ l \wedge invh\ r \wedge bh(l) = bh(r))$

106

Structural invariants

$invh :: 'a\ rbt \Rightarrow bool$

$invh\ \langle \rangle = True$

$invh\ \langle l, _ , _ , r \rangle = (invh\ l \wedge invh\ r \wedge bh(l) = bh(r))$

$bheight :: 'a\ rbt \Rightarrow nat$

106

Structural invariants

$invh :: 'a\ rbt \Rightarrow bool$

$invh\ \langle \rangle = True$

$invh\ \langle l, _ , _ , r \rangle = (invh\ l \wedge invh\ r \wedge bh(l) = bh(r))$

$bheight :: 'a\ rbt \Rightarrow nat$

$bh(\langle \rangle) = 0$

$bh(\langle l, _ , c, _ \rangle) =$

(if $c = Black$ then $bh(l) + 1$ else $bh(l)$)

106

Logarithmic height

Lemma

$rbt\ t \implies h(t) \leq 2 * \log_2 |t|_1$

107

Structural invariants

$invh :: 'a\ rbt \Rightarrow\ bool$

$invh\ \langle \rangle = True$

$invh\ \langle l, _ , _ , r \rangle = (invh\ l \wedge invh\ r \wedge bh(l) = bh(r))$

$bheight :: 'a\ rbt \Rightarrow\ nat$

106

Structural invariants

$rbt :: 'a\ rbt \Rightarrow\ bool$

$rbt\ t = (invc\ t \wedge invh\ t \wedge color\ t = Black)$

$invc :: 'a\ rbt \Rightarrow\ bool$

105

Insertion

$insert :: 'a \Rightarrow 'a\ rbt \Rightarrow 'a\ rbt$

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Insertion

```
insert :: 'a => 'a rbt => 'a rbt
insert x t = paint Black (ins x t)
```

108

Insertion

```
insert :: 'a => 'a rbt => 'a rbt
insert x t = paint Black (ins x t)

ins :: 'a => 'a rbt => 'a rbt
ins x ⟨⟩ = R ⟨⟩ x ⟨⟩
```

108

Insertion

```
insert :: 'a => 'a rbt => 'a rbt
insert x t = paint Black (ins x t)

ins :: 'a => 'a rbt => 'a rbt
ins x ⟨⟩ = R ⟨⟩ x ⟨⟩
ins x (B l a r) = (case cmp x a of
  LT => baliL (ins x l) a r
  | EQ => B l a r
  | GT => baliR l a (ins x r))
```

108

Insertion

```
insert :: 'a => 'a rbt => 'a rbt
insert x t = paint Black (ins x t)

ins :: 'a => 'a rbt => 'a rbt
ins x ⟨⟩ = R ⟨⟩ x ⟨⟩
ins x (B l a r) = (case cmp x a of
  LT => baliL (ins x l) a r
  | EQ => B l a r
  | GT => baliR l a (ins x r))

ins x (R l a r) = (case cmp x a of
  LT => R (ins x l) a r
  | EQ => R l a r
  | GT => R l a (ins x r))
```

108

Adjusting colors

$bal_L, bal_R :: 'a\ rbt \Rightarrow 'a \Rightarrow 'a\ rbt \Rightarrow 'a\ rbt$

109

Adjusting colors

$bal_L, bal_R :: 'a\ rbt \Rightarrow 'a \Rightarrow 'a\ rbt \Rightarrow 'a\ rbt$

- Combine arguments $l\ a\ r$ into tree, ideally $\langle l, a, r \rangle$

109

Adjusting colors

$bal_L, bal_R :: 'a\ rbt \Rightarrow 'a \Rightarrow 'a\ rbt \Rightarrow 'a\ rbt$

- Combine arguments $l\ a\ r$ into tree, ideally $\langle l, a, r \rangle$
- Treat invariant violation **Red-Red** in l/r

$bal_L (R (R\ t_1\ a_1\ t_2)\ a_2\ t_3)\ a_3\ t_4$
 $= R (B\ t_1\ a_1\ t_2)\ a_2\ (B\ t_3\ a_3\ t_4)$

109

Adjusting colors

$bal_L, bal_R :: 'a\ rbt \Rightarrow 'a \Rightarrow 'a\ rbt \Rightarrow 'a\ rbt$

- Combine arguments $l\ a\ r$ into tree, ideally $\langle l, a, r \rangle$
- Treat invariant violation **Red-Red** in l/r

$bal_L (R (R\ t_1\ a_1\ t_2)\ a_2\ t_3)\ a_3\ t_4$
 $= R (B\ t_1\ a_1\ t_2)\ a_2\ (B\ t_3\ a_3\ t_4)$
 $bal_L (R\ t_1\ a_1\ (R\ t_2\ a_2\ t_3))\ a_3\ t_4$
 $= R (B\ t_1\ a_1\ t_2)\ a_2\ (B\ t_3\ a_3\ t_4)$

109

Adjusting colors

$balilL, balirR :: 'a\ rbt \Rightarrow 'a \Rightarrow 'a\ rbt \Rightarrow 'a\ rbt$

- Combine arguments $l\ a\ r$ into tree, ideally $\langle l, a, r \rangle$
- Treat invariant violation **Red-Red** in l/r

$$balilL\ (R\ (R\ t_1\ a_1\ t_2)\ a_2\ t_3)\ a_3\ t_4$$

$$= R\ (B\ t_1\ a_1\ t_2)\ a_2\ (B\ t_3\ a_3\ t_4)$$

$$balilL\ (R\ t_1\ a_1\ (R\ t_2\ a_2\ t_3))\ a_3\ t_4$$

$$= R\ (B\ t_1\ a_1\ t_2)\ a_2\ (B\ t_3\ a_3\ t_4)$$
- Principle: replace **Red-Red** by **Red-Black**

Adjusting colors

$balilL, balirR :: 'a\ rbt \Rightarrow 'a \Rightarrow 'a\ rbt \Rightarrow 'a\ rbt$

- Combine arguments $l\ a\ r$ into tree, ideally $\langle l, a, r \rangle$
- Treat invariant violation **Red-Red** in l/r

$$balilL\ (R\ (R\ t_1\ a_1\ t_2)\ a_2\ t_3)\ a_3\ t_4$$

$$= R\ (B\ t_1\ a_1\ t_2)\ a_2\ (B\ t_3\ a_3\ t_4)$$

$$balilL\ (R\ t_1\ a_1\ (R\ t_2\ a_2\ t_3))\ a_3\ t_4$$

$$= R\ (B\ t_1\ a_1\ t_2)\ a_2\ (B\ t_3\ a_3\ t_4)$$
- Principle: replace **Red-Red** by **Red-Black**
- Final equation:
$$balilL\ l\ a\ r = B\ l\ a\ r$$

Preservation of invariant

After 14 simple lemmas:

Theorem
 $rbt\ t \implies rbt\ (insert\ x\ t)$

Proof in CLRS

Deletion

$delete\ x\ t =\ paint\ Black\ (del\ x\ t)$

Deletion

$delete\ x\ t =\ paint\ Black\ (del\ x\ t)$

```

del _ ⟨⟩ = ⟨⟩
del x ⟨l, a, -, r⟩ =
(case cmp x a of
  LT ⇒
    if l ≠ ⟨⟩ ∧ color l = Black
    then baldL (del x l) a r else R (del x l) a r
  | EQ ⇒ combine l r
  | GT ⇒
    if r ≠ ⟨⟩ ∧ color r = Black
    then baldR l a (del x r) else R l a (del x r)

```

Proof in CLRS

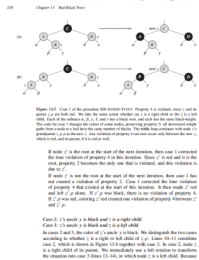
Chapter 17. AVL Trees

The *rotate-left* function in Figure 17.15 maintains the following three-part invariant at the root of each subtree of the tree.

- $h \leq h' + 1$.
- $h \leq h'' + 1$.
- $h \leq h' + h'' + 1$.

where h is the height of the root, h' is the height of the left child, and h'' is the height of the right child.

Figure 17.15 shows the *rotate-left* function. The function *rotate-left* is called on a node x whose left child is y . The function *rotate-left* returns the root of the tree after the rotation. The function *rotate-left* is called on a node x whose left child is y . The function *rotate-left* returns the root of the tree after the rotation. The function *rotate-left* is called on a node x whose left child is y . The function *rotate-left* returns the root of the tree after the rotation.



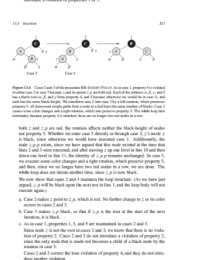
Chapter 17. AVL Trees

The *rotate-right* function in Figure 17.15 maintains the following three-part invariant at the root of each subtree of the tree.

- $h \leq h' + 1$.
- $h \leq h'' + 1$.
- $h \leq h' + h'' + 1$.

where h is the height of the root, h' is the height of the left child, and h'' is the height of the right child.

Figure 17.15 shows the *rotate-right* function. The function *rotate-right* is called on a node x whose right child is y . The function *rotate-right* returns the root of the tree after the rotation. The function *rotate-right* is called on a node x whose right child is y . The function *rotate-right* returns the root of the tree after the rotation.



Deletion

$delete\ x\ t =\ paint\ Black\ (del\ x\ t)$

```

del _ ⟨⟩ = ⟨⟩
del x ⟨l, a, -, r⟩ =
(case cmp x a of
  LT ⇒
    if l ≠ ⟨⟩ ∧ color l = Black
    then baldL (del x l) a r else R (del x l) a r
  | EQ ⇒ combine l r
  | GT ⇒
    if r ≠ ⟨⟩ ∧ color r = Black
    then baldR l a (del x r) else R l a (del x r)

```

Deletion

Tricky functions: *baldL*, *baldR*, *combine*

12 short but tricky to find invariant lemmas with short proofs. The worst:

$$\begin{aligned} & [invh\ t;\ invc\ t] \\ & \implies invh\ (del\ x\ t) \wedge \\ & \quad (color\ t = Red \wedge \\ & \quad bh(del\ x\ t) = bh(t) \wedge invc\ (del\ x\ t) \vee \\ & \quad color\ t = Black \wedge \\ & \quad bh(del\ x\ t) = bh(t) - 1 \wedge invc2\ (del\ x\ t)) \end{aligned}$$

Deletion

Tricky functions: *baldL*, *baldR*, *combine*

12 short but tricky to find invariant lemmas with short proofs. The worst:

$$\begin{aligned} & [invh\ t;\ invc\ t] \\ & \implies invh\ (del\ x\ t) \wedge \\ & \quad (color\ t = Red \wedge \\ & \quad bh(del\ x\ t) = bh(t) \wedge invc\ (del\ x\ t) \vee \\ & \quad color\ t = Black \wedge \\ & \quad bh(del\ x\ t) = bh(t) - 1 \wedge invc2\ (del\ x\ t)) \end{aligned}$$

Theorem

$$rbt\ t \implies rbt\ (delete\ k\ t)$$

Code and proof in CLRS

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Source of code

Insertion:
Okasaki's *Purely Functional Data Structures*

Deletion:
Stefan Kahrs. *Red Black Trees with Types*.
J. Functional Programming. 1996.